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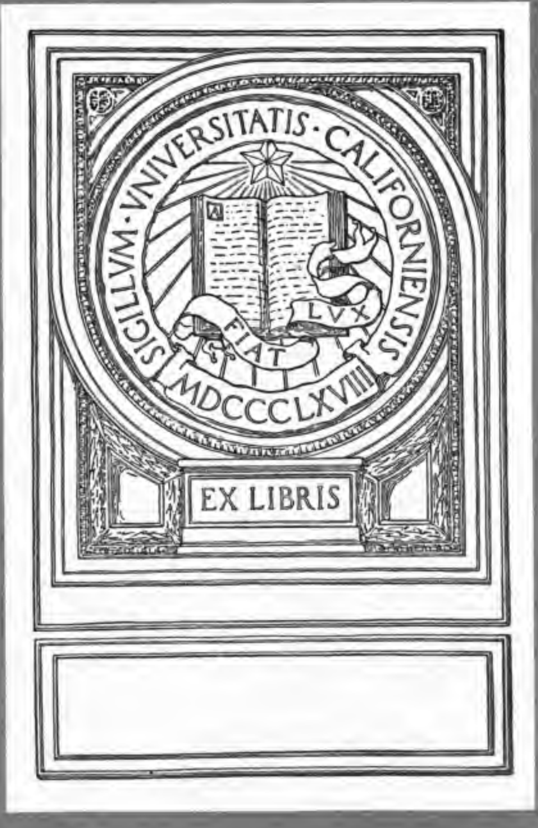

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ELEMENTARY ALGEBRA

PART I.— First Year Course
PART II.— Review Course

PART I

BY

CLARENCE ELMER COMSTOCK, A. M.,
PROFESSOR OF MATHEMATICS
BRADLEY POLYTECHNIC INSTITUTE
PEORIA, ILLINOIS

PEORIA, ILLINOIS
CLARENCE E. COMSTOCK
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PREFACE.

This book is intended as a help to a boy or girl, who is just entering the High School, in gaining that acquaintance with Elementary Algebra which is so essential to success in any future study involving mathematics. Nothing is more certain than that there is a wide difference between the mathematical capabilities of the mind of a pupil at this period, and that of one who is about to enter college. The first year boy is not capable of the abstract thought of his older brother. With this in view the author has divided the course in Elementary Algebra into two parts: PART I (the present volume) A First year course, comprising the simpler portions of the subject, PART II —A Review course with extensions.

Part I can be completed in one year and is sufficient for the needs of work in Geometry and a first course in Physics.

Part II requires from twelve to sixteen weeks and should be given preferably in the last year of the High School. It fully meets the needs of college entrance without encroaching upon the work more properly belonging to college work in Algebra.

In regard to the present volume a number of things should be said at the start.

The aim has been thruout to make the pupil think, and to think for himself. Much that is usually put in a text has been left out, thus giving the pupil an incentive to think, and the teacher the opportunity to guide and correct that thinking.

It will be found that few rules are given and definitions only when they are needed. The pupil is left to form his own rules. The teacher should see that these are eventually put into a form satisfactory to himself and the members of the class, and then carefully recorded in a note book or on the leaves of the book provided for that purpose.

Much work in explanation is left to the teacher, especially at first, for then a subject can often be better comprehended when given by word of mouth, than when read in cold print. Especially is this true with those first two briar patches of Algebra; the use of letters for numbers, and negative numbers. But the pupil must also be taught to glean from the written page an understanding of a subject. This is arranged for in the latter portions of the book.

The order of subjects is not that usually given. Perhaps it may be startling to see the second chapter given to the subject of quadratic equations. But consider that the equation is the real center of interest. The solution of a simple equation is indeed a simple matter when attacked from a common sense point of view. The easily factored quadratic naturally follows.

The first four chapters give a birds-eye view of elementary algebra, using only the simplest material: the simple equation, the quadratic equation, graphic representation and systems of two linear equations. This can be completed in twelve or sixteen weeks. Equations are live with interest. A slight acquaintance with them reveals and emphasizes the need of skill in handling complicated multiplications, fractions and radicals, which in themselves are more or less dry and meaningless to the pupil plunged into them at the beginning of his algebraic study. The more formal work in these subjects is taken up in the later chapters.

Now and then an exercise is thrown in which is beyond the power of the pupil at that stage of the subject. It is given as a spur and as a sign post of the future. It does a boy good to catch a glimpse of the snow capped peak he is to ascend sometime even tho he must be content for the time being to tread the foot hills.

Again, especially in the sections on factoring, unsolvable problems are placed in with the others. In his future work the pupil will not find the functions he deals with nicely sorted and labelled 'factor thus and so'. He must learn to recognize the irreducible ones as well as the factorable ones. A thoughtless pupil may be led to say; "This is another one of those examples that cannot be done." But he will soon learn the wisdom of carefully scrutinizing all work to be done which habit is the key to algebraic success.

Much emphasis is laid upon checking results. This not only secures for the pupil certainty in his results but also affords excellent opportunity for practice in reckoning. The verification of fractional and irrational roots of equations give rise to the best exercises in fractions and radicals.

The effort has been made to make the work direct and purposeful. Gymnastic work for the mere sake of doing something is largely omitted. The exercises are for the most part such as one would be called upon to do in the serious study of mathematics. For illustration note the treatment of radicals.

The problems selected for statement are in general practical problems such as arise in the serious study of mathematics and science. It is thus hoped to impress the pupil with the importance and value of algebra in the life of the world.

The graph has not been used as a side light and after thought but as an integral part of the subject. It has been given an early place. It has seemed best to lead up to the graph as a representation of the solution of an equation thru the consideration of the graphic representation of a series of numbers unrelated by a known law. The solution of a system of equations is a tangled briar patch wrapt in almost stygian blackness without the graph. The pupil may learn to follow a guide rope but he is blind to all the meaning of it.

Ratio is considered briefly in the last part of the chapter on fractions. The idea of variation is used wherever it may be of service.

The endeavor has been to bring the pupil into contact with a real living subject, and it is to be hoped that when he has completed this volume he will realize that he has made but a beginning in the study of algebra.

CLARENCE E. COMSTOCK.

Bradley Polytechnic Institute.
April 1907.

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DEFINITIONS AND RULES.

(TO BE FILLED IN BY PUPIL.)

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UNIV. OF CALIFORNIA

ELEMENTARY ALGEBRA

PART I

FIRST YEAR COURSE

CHAPTER I.

SIMPLE EQUATIONS.

1. 1. Find the area in square feet of the top of your desk. State the result as neatly as you can and give your reasons for thinking that your result is correct. On coordinate paper show that your way of getting the area is right.

Find the area of the front cover of your Algebra, the glass in the door, the glass in the window, the top of the teacher's desk, face of the cube on the desk, the strip of cardboard furnished you, and state your results as before.

2. Draw a diagram which will represent the kind of a thing you have been measuring. What do you call such a figure? What are its characteristics? What is a right angle? Take a sheet of note paper and show that the angles at the corners are all equal and that the opposite edges are equal. What do you mean by equal? State as briefly as you can your rule for finding the area of a rectangle.

The rule you have just given states a relation which exists between three numbers, the area, length and breadth of a rectangle. We often speak of such a relation as a law connecting the numbers.

The law for the area of a rectangle is then:

The area of a rectangle equals the length of the rectangle times its breadth. Or briefly:

Area equals length times breadth. Or still more briefly:

$$a = l \times b$$

We may even omit the times sign and write

$$a = lb$$

understanding that the numbers for which l, b stand are to be multiplied together.

(You remember that you have made a similar omission of sign in connection with fractions, putting $2\frac{3}{8}$ for $2 + \frac{3}{8}$)

The law first stated in English words, has thus been translated into what we call Algebraic notation (language). This algebraic language is a sort of shorthand. It has been growing thru the ages just as has the English language and it is still growing. For instance the sign = was used first by a man by the name of Recorde in 1540, fifty years after the discovery of America by Columbus: the signs +, — are due to Rudolf in 1524: other symbols which we will use have been introduced in the last few years.

3. From the strip of cardboard cut a rectangular piece having an area of 12 square inches.

4. The length of a rectangular field is 25rds. and its area 475 sq. rds: what is its width?

Carefully state your work for the last two examples.

But notice the following method of working the fourth:

If b = the breadth of the field

then $25b = 475$

$b = 475/25$

$b = 19$

(475/25 is a better way of expressing division than $475 \div 25$)

It is easily seen that the equation $a = lb$ can be used in finding any one of the three numbers if the other two are given. What is an equation?

In solving the following problems state your work in some such form as above.

5. The depth of a block from street to alley is 150 feet: what must be the frontage of a lot which is to contain one half an acre of ground?

6. A train travels at the rate of 40 miles an hour; how far does it go in 2 hours? in 5 hours? in $\frac{3}{4}$ hours?

7. The Knickerbocker Express leaves Peoria at 12 n'n and reaches Cincinnati at 9:05 p. m.: what is its average rate of speed, the distance being 322 miles?

8. If an automobile is going at the average rate of 15 miles an hour, how long will it take to go 35 miles?

The form in which we have written the work of the foregoing problems appears to be an improvement upon that we have been accustomed to in our arithmetic work. This orderly arrangement is one of the features of algebra. You will notice also that we have considered two different kinds of problems, one about rectangles, the other about what is called uniform motion. In each case we have stated in the form of an equation the law connecting the numbers involved. By the aid of this law we were enabled to solve the various cases coming under each sort of problem. You remember what a time you had to remember the various cases in percentage and interest. Let us see if the equation does not simplify matters.

9. State the law of percentage using B. r. p. for base, rate, and percentage.

Now solve.

10. What is 5% of 70?

✓ 11. What per cent. of 65 is 26?

12. $\frac{3}{5}$ is what per cent. of $\frac{4}{5}$?

13. 24 is 6% of what number?

You see you can now make your own rules for the various cases. For they are all packed up in the one rule, the equation, as in a nutshell.

Try this one:

14. A lot is sold for \$960 which is 20% more than it cost. Find the cost.

15. Place a number of weights in one scale pan of a balance and balance with tenpenny nails in the other pan; what can you say of the weights of the articles in the two pans? Write your answer in English and then in algebraic language.

Write an equation involving the number of nails and the number of weights.

Knowing the weight of each of the brass weights, can you find the weight of the nails? of each nail? Express your results carefully.

16. Nine times a number is thirty-five. State this sentence in the form of an equation expressed in algebraic symbols. And then find the number.

17. What number added to seven times itself equals 104?

18. Seven times a number less four times the number is 123, what is the number?

2. In all the problems we have just worked, notice the equation and how we use it. For instance in the 17th:

If n = the number

$$7n + n = 104 \qquad (1)$$

$$8n = 104 \qquad (2)$$

$$n = 13 \qquad (3)$$

Take equation (1) which is the statement of the facts given in the problem.

7n means 7 times the number.

So we may regard n as meaning one times the number.

We add 7n and n together and so get equation (2). To get equation (3) we divide both sides of equation (2) by 8, and of course the results are equal for every one knows that if you divide equal numbers by the same number the results are equal.

It follows that one can always divide both sides of an equation by the same number.

This can be considered one of the properties of the equation.

The two sides of the equation are called the members of the equation.

In the product $7n$ either factor is called the coefficient of the other.

7 is called the numerical coefficient.

The unknown number you are asked to find is called the root of the equation and the finding of it is called the solution of the equation.

The work may be tested by substituting the root in the equation, *i.e.* by replacing the letter by the number found. The solution is correct if the two members can be reduced to the same form.

When the two members of an equation are exactly alike it is called an identity.

Solve and check results

19. For what value of s is $10s + 8s - 4s = 42$.

20. For what value of n is $13n - 8n = 2$.

21. $7t - 3t + \frac{1}{2}t = \frac{1}{9}$.

22. $8x - 8x + 3x - 1\frac{1}{3}x = \frac{2}{3} + \frac{1}{2}$.

23. $0.8x - 0.25x = .165$.

3. 24. What number added to 13 equals 29?

Notice this way of writing the work:

If n equals the number

$$\text{then}\qquad \begin{aligned} 13 + n &= 29 \\ n &= 29 - 13 \\ n &= 16 \end{aligned}$$

25. What number would you put in place of b to make

$7 + b = 15$ true?

26. How far is it around the front cover of your algebra? State your work in good form. State the law connecting the length, breadth and perimeter of a rectangular figure. If any two of these numbers are given can the other be found?

27. How much wire fencing will it take to fence a lot 60 by 150 feet?

28. From the strip of paper furnished you cut off a rectangular piece having a perimeter of fourteen and a half inches. Test your work by actual measurement.

29. It takes 380 feet of wire fencing to enclose a lot which has a depth of 150 feet; how wide is the lot?

30. A rectangular field of which the length is three times the breadth requires 850 feet of fence to enclose it; what are its dimensions?

31. It takes 280 feet of wire fence to enclose a square lot; what are the dimensions of the lot?

32. Add 45 to four times a number and you will have seven times the number; what is the number?

What has been done to each member of the equation in solving each of the last nine problems?

State this as the second property of the equation.

Solve and check the following:

33. $x + 18 = 23$

34. $100 = 92 + t$

35. $19 + n = 25$

36. $30 - 15 = 4t + 5$

37. $5r = 3r + 7$

38. $5n + 2 = 3n + 4$

39. $2 + 11p + 3 = 2p + 15$

40. $121A + 2 = 10 + 7A - 5 + 17A$

41. $\frac{4}{5}x - 2 = \frac{1}{2}x + \frac{1}{5}$

4. 42. Five times a number less 17 equals 8; what is the number.

43. Nine times a number less 27 equals six times the number; what is the number?

In solving these last two equations what must be done to the equations that you have not done in the other cases?

State this as a third property of the equation.

Find the roots and check.

44. $x - 15 = 30$ = 45
45. $3t - 5 = 13$ = 6
46. $4 - x = 2$ = 2
47. $n - 7 - 6 = 3$ = 14
48. $7t - 3 = 8$
49. $8n - 1 = 5n + 41$
50. $3x - 21 = 2x - 11$
51. $2h = 5h - 5 - 2h$
52. $2T - 3 = 3T - 4$
53. $11 - 5t + 17 - 3t = 18 - 11t + 23$
54. $\frac{1}{2}y - \frac{3}{4} = \frac{1}{3}y - \frac{1}{3}$

5. 56. One third of a number is 11; what is the number?

57. One seventh of a number is 5; what is the number?

To get the answer to the 56th you multiply 11 by 3, or stating in algebraic language;

If n = the required number

$\frac{1}{3}n = 11,$ $n = 11 \div 1/3$ by property. 1

$n = 33$

Can you think of another way of solving the same equation? (Notice the remark made just after the 57th problem.)

Which do you think the better way? Try both methods on this example:

One third of a number plus one half the number equals five: what is the number?

Which seems to you the better way? Why?

Thus we are led to a fourth property of the equation. State it please.

Use this property in the solution of the following equations:

58. Find the value of n in $2 = n/5$

59. Show that $\frac{a}{3}$ or $a/3$ means the same as $\frac{1}{3}a$

60. $\frac{5 - m}{2} = 1$

61. $\frac{2}{3} = \frac{a}{5}$

62. One fifth of a number plus one seventh of the number equals 11; what is the number?

63. Here are two rectangular blocks, one of oak, one of lead of about the same size, but one is much the heavier. Find the weight and volume of each in grams and centimeters.

The weight of a unit volume of a substance is called the density of the substance.

Find the density of the substances just measured. Write an equation showing the relation between the three numbers, weight, volume and density.

64. Find the density of this rectangular piece of glass.

65. If the density of granite is 2.65 what will be the weight of granite block of dimensions 2 by $1\frac{1}{2}$ by 1 meters.

66. Find the volume of this irregular piece of oak, and of this chunk of lead using the density found in the previous problem.

67. The density of gold is 19.3, and that of aluminum is 2.6: what will be the weights of blocks of gold and aluminum the size of the blocks of oak just used?

68. $m - 3m/4 = 8$

69. $t/12 + t/6 = 5$

70. $1/4 + 1/6 = 1/F$

71. $1/40 + 1/D = 1/10$

72. $3a/4 - 2a/2 = 11$

73. $\frac{4}{3}x - 2 = \frac{1}{2}x + \frac{1}{3}$

74. $\frac{1}{2}y - \frac{3}{4} = \frac{1}{3}y - \frac{1}{3}$ Compare with Ex. 54.

✓ 75. The sum of two numbers is 90 and the smaller is one fifth of the larger; what are the numbers?

6. In solving the foregoing problems we have made use of four important properties of the equation. They are often called axioms and may be stated thus:

Axiom 1.—If equal numbers be added to equal numbers, the sums are equal.

Axiom 2.—If equal numbers are subtracted from equal numbers, the remainders are equal.

Axiom 3.—If equal numbers are multiplied by equal numbers, the products are equal.

Axiom 4.—If equal numbers are divided by equal numbers, the quotients are equal.

The four may be packed into one statement:

If the same arithmetical operation be performed on both members of an equation the results will be equal.

In solving an equation we must get the unknown number, alone, on one side of the equation. Any device that will help in doing this may be used if it does not run counter to the axioms stated above.

State a rule for solving equation keeping in mind the various styles of equations we have solved thus far.

7. Find the roots of the following equations and after each step state the axiom used and check all results.

76. $x - 21 = 0.$

77. $6r - 5 = 3r + 10.$

78. $6a - 8 = 3a + 13$.

79. $5 + (3m - 4) - m + 8 = 0$.

80. $13(4y + 3) = y + 2(y + 5) + 29$.

81. $7 - x = 3(4 - x) + 1$.

82. $\frac{7}{8}(24t - 800) = \frac{2}{5}(45 - 5t) - 5$.

83. $k/4 + k/2 - k/5 = 11$..

In the problems on density the decimal numbers given were approximate, they all come from measurements which you know of course are never exact. When such numbers are used in equations, the root of the equation should also be found in the decimal form, which of course will be approximate. When these values are substituted the two members will not be exactly equal, but if the work has been done correctly the difference will be slight.

In solving problems where measurement is involved this should always be remembered.

84. $32.53n = 7.237$.

85. $0.023x + 2.583 = 9.357 + 0.003x$.

86. $2.34p + 0.75p - 3.02 = 8.14 + 1.83p$.

87. $x + 21 = 5$.

8. In the last example you met with a difficulty, for you could not subtract 21 from 5. What shall we do about it? Give it up? or wait for more light? The man who wants to know things and who wants to do something never gives up until he absolutely knows he can go no further. He rather searches around for more light even if it takes him years.

9. Negative Numbers.—In your study of arithmetic you have come across two kinds of numbers, integers and fractions. Do you know whether there are any other kinds of number?

7 less 3 is what?

3 less 7 is what?

Is it ever possible to take a larger number from a smaller? Let us see.

You cannot take five apples from a plate that contains but three, but you can figure out how much you are worth in money when you buy a three dollar hat with only one dollar to your name. You have a debt of two dollars.

If the temperature of a room is 10° can you take away 15 degrees of temperature? How would you express the resulting temperature?

To a balloon which has an upward pull of 515 pounds is attached an anchor which weighs 390 pounds: how can you express the weight of the anchor, of the balloon, of the two combined?

(Note to teacher.—Illustrate by the use of the negative number apparatus.)

All these cases show the need of another kind of number to express all the kinds of things we wish to: integers and fractions are not sufficient. We are thus led to a new kind of number called negative number. We indicate such numbers by writing before them the minus sign, —.

Thus —7 is called a negative number.

7 or +7 is now called a positive number.

— and + are called the signs of the number.

The numerical value of the number regardless of the sign is called the absolute value of the number.

We are now ready to answer the question: Can we subtract a larger number from a smaller one?

When we subtract 5 from 8 we count off five from 8. Counting off one at a time we have three left. Now when we subtract 8 from 5 we count off 8 in the same way; but when we have counted off 5 the minuend is all gone and 3 is left to count off. This can be indicated thus:

$$-3, \text{ i.e., } 5-8=-3.$$

$$\begin{aligned}
5-1&=4.\\
5-2&=3.\\
5-3&=2.\\
5-4&=1.\\
5-5&=0.\\
5-6&=-1.\\
5-7&=-2.\\
5-8&=-3.\\
5-9&=-4.
\end{aligned}$$

and so on; and we have a series of numbers below 0 just like the series above 0 except that they are all negative numbers; and they are smaller as you go down.

We may now write these numbers in the following way:

$$-4, -3, -2, -1, 0, 1, 2, 3.$$

Where in this series of numbers would you place the numbers:

$$3\tfrac{1}{2}, -3\tfrac{1}{2}, \tfrac{2}{3}, -\tfrac{3}{4}, -5, -10?$$

As has been said these new numbers will be found to be very useful. They were probably invented first by the Hindus.

10. We can speak of a temperature of —10° but not of a box —10 inches square. Why not?

Give other instances in which negative numbers may or may not be used.

If we are to make use of negative numbers we must know how to add, subtract, multiply and divide them.

Suppose we denote a pull down (weight) by positive numbers and a pull up by negative numbers, thus,

+50 grams means a pull of 50 down.
—50 grams means a pull of 50 up.

(The negative number apparatus will be of use here.)

It is easy enough to add two positive numbers, such as,

10 grams and 50 grams;

or two negative numbers, such as,

— 10 grams and — 50 grams.

What, however, is the sum of

+50 grams and —20 grams;

+20 grams and — 50 grams?

Smith's assets are $3,450.00 and his liabilities are $5,000.00; how much is he worth?

We may now state the Laws of Sign for Addition:

1. The sum of two numbers which have the same sign is the sum of their absolute values preceded by the common sign:

$$(-5)+(-2)=-7.$$

2. The sum of two numbers which have different signs is the difference of their absolute values preceded by the sign of the number having the larger absolute value:

$$(-5)+(+2)=-3.$$

(The class will now be given considerable practice in the addition of positive and negative numbers supplied by the teacher.)

88. If we regard motion to the north as positive and motion to the south as negative, where is a bicyclist with respect to his starting point if he rides north 5 blocks, south 7, north 3, south 8, north 4, south 2?

89. Compute the average temperature from noon Dec. 25 to noon Dec. 26, 1903, given the hourly readings:

28, 12, 8, 4, 0, —2, —5, —7, —6, 0, 5.

90. What is the sum of 3x, — 2x, 7x, — 25x, — 15x, + 28x, —4x, —27x?

91. The 90th may be written

$$3x-2x+7x-25x-15x+28x-4x-27x$$ 37 -

and then simplified. Why?

11. We have now to consider the subtraction of negative numbers.

1. Francois Vieta, the first man to write ab for $a \times b$, was born in 1540 and died in 1603: how old was he when he died?

2. Augustus Caesar was born in -63 and died in 14: how old was he when he died?

3. Pythagoras was born in -580 and died in -501; how old was he when he died?

By common sense arithmetic you get your results for the last three examples, thus:

For 1) $1603 - 1540 = 63.$

2) $63 + 14 = 78.$

3) $580 - 501 = 79.$

Each after its own pattern. Now in reality they can all be solved after one pattern. The law involved may be stated thus:

age = date of death — date of birth.

$$a = d - b.$$

$$63 = 1603 - 1540.$$

$$78 = 14 - (-63).$$

$$79 = (-501) - (-580).$$

It will be noticed that in each case we have found a number which added to the subtrahend gives the minuend.

Subtraction is the inverse of addition.

In addition we ask $5 + 3 = ?$

In subtraction we ask $5 + ? = 8.$

(or as it is usually written $8 - 5 = ?$ *ie.*, we ask what number added to 5 gives 8? Evidently 3.

So also $(-3) + ? = 5$; evidently $8.$

$(-8) + ? = -3$; evidently $5.$

$(-5) + ? = -7$; evidently $-2.$

$9 + ? = 4$; evidently $-5.$

If we arrange the numbers in order, thus,

$-9, -8, -7, -6, -5, -4, -3, -2, -1, 0; 1, 2, 3, 4, 5, 6, 7, 8;$

we notice that to get the number to be added, when the minuend number is greater than the subtrahend number, we have to count to the right, and get a positive number as a result.

When the minuend number is less than the subtrahend number we count to the left and get a negative number as a result.

(Let the class practice a little with this devise.)

Altho we can always get the results of subtraction in this way we desire a better and quicker way.

Using the usual subtraction notation we write

$$
\begin{aligned}
8 - 5 &= 3 \\
5 - (-3) &= 8 \\
(-3) - (-8) &= 5 \\
(-7) - (-5) &= -2 \\
4 - 9 &= -5
\end{aligned}
$$

These same results may be gotten thus:

$$
\begin{aligned}
8 - 5 &= 3 \\
5 + 3 &= 8 \\
-3 + 8 &= 5 \\
-7 + 5 &= -2 \\
4 - 9 &= -5
\end{aligned}
$$

(These results are best shown on the negative number apparatus.)

The laws of sign for subtraction can be stated thus:—

The subtraction of a number gives the same result as the addition of a number of contrary sign having the same absolute value.

Subtraction, then, can always be replaced by addition if care be taken to change the sign of that which is to be subtracted.

A restatement of the laws of sign for addition and subtraction is

$b + (+a) = b + a \qquad b - (+a) = b - a$

$b + (-a) = b - a \qquad b - (-a) = b + a$

$b + (a - c) = b + a - c \qquad b - (a - c) = b - a + c$

(Considerable practice in subtracting positive and negative numbers should now be given.)

12. Consider now the multiplication and division of negative numbers.

State your definition of multiplication and in accord with that definition perform the following multiplications:—

$2 \text{ times } 7 = ?$

$2 \times 8 = ?$

$(2) \times (-7) = ?$

$(-3) \times 4 = ?$

$(-3) + (-4) = ?$

The last one of the five requires a new definition of multiplication, which we can get at thus:—

4 is the sum of four 1's $\qquad 4 = 1 + 1 + 1 + 1$

4×3 is the sum of four 3's, $\qquad 4 \times 3 = 3 + 3 + 3 + 3 = 12$

so $\quad 3 = 1 + 1 + 1 \quad$ hence $\quad 3 \times 1/2 = 1/2 + 1/2 + 1/2$

and $\quad 3/5 = (1 + 1 + 1) \div 5$

hence $3/5 \times 2 = (2 + 2 + 2) \div 5 = 6/5$

and again $\quad$ as $2/3 = (1 + 1)/3$

so $2/3 \times 3/5 = (3/5 + 3/5)/3 = 2/5$

The new meaning given to multiplication gives us the definition:—

The product of any two numbers is the result of performing upon the multiplicand the same operation that must be performed upon unity to obtain the multiplier.

Applying this to the case of negative numbers:

since $-3 = -(1+1+1)$

it follows that $(-3) \quad (+4) = -(4+4+4) = -12$

and $(-3) \quad -4) = -(-4-4-4) = -12$

Our results are

$$(+3) \quad (+4) = +12$$
$$(+3) \quad (-4) = -12$$
$$(-3) \quad (+4) = -12$$
$$(-3) \quad (-4) = +12$$

Can you derive the Law of Signs for multiplication? State in words.

13. Division is the inverse of multiplication. Derive the Law of Signs for division.

Laws of Signs for multiplication and division.

The product or quotient of two numbers of like sign is a positive number.

The product or quotient of two numbers of unlike sign is a negative number.

(Considerable practice should be given in the multiplication and division of positive and negative numbers. A thoro mastery of the laws of sign for the four operations will save you from many errors. There should be no hesitation in your mind as to the sign in any case that comes up.)

Now that we have manufactured a little more algebraic machinery we are ready to proceed in our investigations on equations.

We can solve Ex. 87: $x + 21 = 5$

Solve and check

92. $x - 5 = 0$

93. $x + 5 = 0$

94. $7m + 13 = 4m$

✓ 95. $9n - 3n + 4n + 7 - 3 = 0$

✓ 96. $4r + 5 = 5(r + 1)$

97. $5 + x = x/2 - 5$

98. $7 - (3n - 15) = 4.$

14. By the use of axioms 1 and 2 we get

(1) $3x - 5 = 7 + x$
(2) $3x - 5 + 5 - x = 7 + x + 5 - x$
(3) $3x - x = 7 + 5$

This work may be shortened. We may get (3) from (1) by removing -5 to right side, at the same time changing its sign, and by removing $+x$ to the left side, changing its sign from + to —

This is called transposition. That this could be done was discovered by the Arabs, and it is interesting to notice that it is this transposition that gives the name algebra to the branch of mathematics we are studying.

Sometime between 813—833, an Arab by the name Mohammed ibn Musa Alchwarizmi wrote a book on mathematics, the title of which is "Aldschebr Walmukabala." The first word was the word he used to mean transposing a negative number from one side of the equation to the other. It is our word algebra, tho with a rather more complicated spelling.

99. $\frac{x - 32}{9} = \frac{x}{5}$

100. $50 - 2x + (x - 10)3 = 0.$

101. $5(y + 1) - 3(y + 5) = 2(6 - y) - 2.$

102. $-12 + \frac{3}{5}T = 0.$

103. $4/m = 5/m + 1.$

104. $p/5 + p/7 = 46 + p.$

105. $28 - 25(.4c + .2c) = 2c - 12.$

106. $\frac{7 - 3(x - 5)}{4} = 1$

107. $3 - \frac{n+2}{4} = 2$

108. $\frac{c-2}{2} - \frac{c+2}{5} - \frac{c-1}{7} = 0.$

109. What number divided by 12 gives the same result as when added to 12?

110. The first side of a triangle is 1 inch longer than the second side, the second is 7 inches longer than the third and the perimeter is 12 inches. What are the dimensions of the triangle?

111. One side of a triangle is two inches longer than a second, and the second is one inch longer than the third. The perimeter is 19 inches, what are the dimensions of the triangle?

PROBLEMS AND EXERCISES.

15. 112. What is the area of the surface of the rectangular block of wood on your desk? Tabulate your work in good form and state the results in the form of an equation.

Write an equation which will be true for all rectangular solids.

113. How far from the end of this rod must I cut in order to cut off a block which shall have a surface area of 37 square inches.

Test by measuring the block cut off.

114. One side of a triangle is 2 inches longer than the second side and the second side is 3 inches longer than the third side and the perimeter is 29 inches; what are the lengths of the sides? Draw the triangle.

If the student cannot draw the triangle show how this can be done by the use of a compass.

115. The base of a triangle is 5, its altitude is 3; find its area.

Show in some way that your result is correct.

State a formula for the area of any triangle.

116. The area of a triangle is 35 sq. in., its base is 7 inches; what is its altitude? Draw the triangle reducing the dimensions one half. What is the area of the triangle drawn?

117. A train moves at the average rate of 30 miles an hour, how far does it go in 6 hours?

118. If a body moves at the same rate all the time the motion is said to be uniform.

Letting $v =$ velocity, $t =$ time, $s =$ distance, or space passed over. State the law of uniform motion connecting these three numbers.

When t equals unity what does s equal?

119. How long will it take a train to go one block of 300 feet if it were running at the rate of 30 miles an hour?

120. Assuming the velocity of sound to be 1100 feet a second, find the distance of the point of discharge if 24 seconds elapse between the time of seeing the lightning and the time of hearing the thunder.

(Try one or two cases with stop watch or metronome, signals being given by the students.)

121. The velocity of sound thru the air is not constant for all temperatures. The following formula has been found, by means of which the velocity can be determined for any temperature.

In the formula F denotes the temperature on the Fahrenheit scale.

$$v = 1084.8\sqrt{1 + 0.0021\ (F - 32)}.$$

What is the velocity at 0°, 32°, 60°, 100°?

In case the student does not know how to extract the square root he should be shown the usual method. The theory of the method will be considered later.

122. Dense fogs often make sailing along the fjords of Iceland dangerous. The land rises abruptly from the sea and the echoes from the sheer faces of the rock enable the officers of a vessel to judge of their distance from shore.

How can this be done? In one case the echo came 4 seconds after the whistle sounded; what was the distance from shore?

123. Divide the number 96 into three parts such that the first exceeds the second by 5 and the second is three times the third.

124. If p stands for principal, t for time in years and r for rate per cent, find i (interest) and a (amount) in terms of p, t and r.

125. Apply to following cases:—

Principal, \$9000, rate 6%, interest \$632; find time.
Rate 5%, time 3 yrs., interest \$22.50; find principal.
Amount \$97.57, rate 4%, interest \$7.57; find time.
Principal \$135, time 4 mos., amount \$137.25; find rate.

16. 126. What number added to 9 times itself equals 70?

127. What number added to a times itself equals b?

The first of these two problems deals with a special case. The second is more general and includes hundreds of thousands of special cases like the first.

So in solving the second we have solved all the others. All that would be necessary in any given case would be to substitute the values in our general result.

$$n = \frac{b}{1+a}$$

Note that in the equation

$$n + an = b$$

n is regarded as the unknown number.

We add n and an by adding their coefficients $(1+a)\,n = b$.

and then $n = \dfrac{b}{1+a}$

Where letters are used for numbers we can only express the operations to be performed.

Thus

The sum of two numbers: $a + b$

One-half their difference: $\frac{1}{2}(a - b)$ or $\frac{a - b}{2}$

Their quotient $\frac{a}{b}$

These general expressions which will apply to any special cases when desired, comprise one of the great features of algebra. By means of them we can solve general problems thus disposing of hosts of special cases at once and for all.

A good case in point is the special problem 128.

128. The sum of two numbers is 160 and their difference is 40; find the numbers.

Generalize, use s for sum,. d for difference. Observe that the result may be used as a formula.

129. Make two problems to fit the equation

$$ax + bx = c$$

130. Make two problems to fit the equation

$$ax = b$$

17. We have seen that equations express a law connecting certain numbers involved in the problems under discussion.

For example.

$A = lb$	the area law of a rectangle.
$p = Br$	percentage law.
$D = \frac{w}{v}$	Density law.
$p = 2(l + b)$	Perimeter law of rectangle.

In any of these cases if the values of all but one of the numbers involved are known, the remaining one may be easily found if only you handle the equation in the right way.

We have done this in special cases. It may be done in general just as well.

If we fix upon the number that is unknown we have then but to so handle the equation that that number turns up alone on one side of the equation while the other side contains none but known numbers.

The equation is said to be solved for that letter (number). An equation can be solved for any letter.

131. $w = fs$ solve for s, and for f

132. $C = E/R$ solve for E, and for R

133. $W + w = nP$ solve for P, and for w

134. $w/F = 1/h$. solve for each letter

135. $a = bx + c$ solve for x

136. $d = 2i - A$. solve for i

137. $A + (B - T) + (C - T) = \frac{1}{2} s$. Find T.

138. $\frac{1}{M} - \frac{1}{E} = \frac{1}{S}$ solve for M and find value of M when $E = 365.25$, $S = 29.53$

139. $\frac{H}{v} = 20\frac{t}{n}$ solve for n

also determine the value of v when

$$H = 3, \quad t = \frac{1}{2}, \quad n = 50.$$

140. $wa + pc - pb = 0$ solve for p.

141. $ax - a = bx - b$ solve for x.

✓142. $a(x - b) = b(x - a)$ solve for x.

CHAPTER 2.

QUADRATIC EQUATIONS.

18. The area of a rectangle is given by the formula $A = lb$. Hence the area of a square will be $A = bb$; or, as it is better written, $A = b^2$.

You are familiar no doubt, with the notation b^2.

The number written to the right a little above the letter is called an exponent.

When such a number is a positive integer it indicates that the letter which it effects is to be used that many times as a factor.

2^4 means $2 \times 2 \times 2 \times 2$.
a^5 means a.a.a.a.a..
a^5 is called the fifth power of a.
a^2 is called the second power of a, or the square of a.

19. 1. If the side of a square is 8 inches what is its area?

2. If the area of a square is 81 square inches what is its side?

Writing this in algebraic language we have:—

$$81 = b^2$$
$$\sqrt{81} = b$$
$$9 = b$$

9 is the root of the equation.

Is 9 the only number that will satisfy the equation? Try —9. It appears that this equation has two roots.

Will both these roots serve as answers to the problem?

From this it appears that 81 has two different square roots, 9 and —9, for $9 \times 9 = 81$ and $(-9) \times (-9) = 81$. (Note the difference between the meaning of the two phrases: root of an equation, and square root of a number).

What are the square roots of 25; 64; 625; 36?

The fact that every number has two square roots of the same absolute value but differing in sign is very important and must always be taken into account when one is extracting a square root.

20. Equations which contain no higher power than the square of the unknown number are called quadratic equations.

Quadratic equations of the form $x^2 = 81$ can be solved by extracting the square root of each member, care being taken to use both signs for the square roots of the known number.

What are the characteristics of the form given above by which you can recognize it?

3. Wrap a strip of paper around a cylinder and fasten with a pin. Then straighten out the paper and measure the distance between the two pinholes. Also measure the diameter of the cylinder.

Change cylinders with your neighbor and make the same measurements again.

Make the same measurements on your pencil.

Tabulate the results in two columns marked:—circumference, diameter, circum./diam.

Compute the ratio of the circumference to the diameter and place results in the third column.

What seems to be true about this ratio?

State your conclusion in algebraic form. Of what value is this statement?

This constant ratio of the circumference to the diameter of a circle is generally denoted, in short, by the Greek letter π, the first letter of the Greek word perimetron, meaning circumference. It is a very peculiar number: its value cannot be exactly expressed in numbers, tho we may get it approximately as close as we choose.

In your arithmetic work you have probably used 3.1416. 3.14159 is a little closer. King Solomon used 3.

Its value has been computed out to 800 decimal places, tho no one would have any need for such an approximation, $3\frac{1}{7}$ or $\frac{22}{7}$ is a value close enuf for our use.

Find how this approximation compares with the value 3.14159.

4. The radius of earth is about 3,960 miles, what is the distance around the earth?

5. What is the circumference of a one cent piece? Find by measuring the diameter.

6. One of the largest trees in Illinois is a bald cypress in Pulaski County: it was found to measure 21.6 feet in girth, four feet from the ground. What is its diameter?

(It will probably be necessary to discuss with the class the uselessness of the string of decimals the most of the class will have in their results for the examples just given.

7. What is the diameter of the steam pipe in the corner of the room?

8. Find the area of the circular card furnished you.

If $a =$ area and $r =$ radius, the formula for the area of a circle may be written $a = \pi r^2$.

9. Find the area of a circle whose radius is three feet.

10. Find the area of a circle whose diameter is 4 feet.

11. Find the radius of a circle whose area is 64 square feet.

12. What is the radius of a circular field containing one acre?

(It is of course, expected that all members of the class can extract the square root of a number.)

13. Draw a circle which shall have an area of 5 square inches.

Does the circle have exactly that area? Why?

Find the roots and check each of the following:

14. $x^2 = 25$.

15. $9x^2 = 49$.

16. $75t^2 = 27$.

17. $98 - 8t^2 = 0$.

State a rule for solving quadratic equations.

18. $225 - 3x^2 = 144 + 4x^2$.

19. $3s^2 - 41 = 39 - 2s^2$.

20. $3n^2 - 15 = 0$.

21. $15 = \frac{4}{3} P^2$.

22. $924 = r^2 + (9 - r)^2 + r(18 - r)$.

23. $11 (11 + d) (11 - d) = 792$.

21. 24. What is the side of a square whose area and perimeter are expressed by the same number?

You may be able to guess at the solution and hit the right answer, but how do you get your result out of the equation? Can you solve this quadratic in the same way you have solved the others? We must have a new method. Can you see how to get out of the difficulty?

25. Try this equation: $9s^2 = 4s$.

Your method of division gives you a correct result, but it does not tell the whole story. There is another root of the equation. Can you find it.

22. We must do a little skirmishing to find the new method.

We have seen that by adding we get

$$3x + 5x = (3 + 5)x = 7x.$$
$$ax + bx = (a + b)x.$$

Let us look at this in a different light:

The first member $ax + bx$ is the sum of two numbers, while the second $(a + b)x$ is the product of two numbers. We say that we have factored the expression $ax + bx$.

In the same we can factor any expression that has the same letter or number in each term.

Thus $ac + bd$ cannot be factored, but $xx + ax = (x + a)\,x$, *i.e.*, $x^2 + ax = (x + a)\,x$.

Factor $x^2 - 18x$, $t^2 - 9t$, $7n^2 + 14n$.

23. Again $0 \times 3 = ?$, $7 \times 0 = ?$ $0 \times a = ?$

If a factor is zero the product is zero.

If a product is zero and one of the factors is a, what is the other factor?

If a product is zero what must one of the factors be?

The just noticed fact may be stated thus:—

The Factor Axiom.—(1.) If a factor equals zero, the product must equal zero; and (2.) If a product equals zero, at least one of its factors must equal zero.

This may be put: Let $ab = n$.

If $b = 0$, then $n = 0$,

If $n = 0$, then $a = 0$, or $b = 0$ or both $= 0$.

24. Now we are ready to solve the equation $x^4 = 4x$.

Transposing, $x^2 - 4x = 0$.

Factoring, $x\,(x - 4) = 0$.

Here is a product equal to zero. That can be true only when one of its factors is zero. The equation is satisfied then if either of the two factors be put equal to zero.

The first factor is zero when $x = 0$.

The second factor is zero when $x = 4$, therefore the roots are 0 and 4.

26. Find a number such that two times its square is fourteen times the number.

State the new method in the form of a rule.

Solve and check.

27. $3x^2 + 9x = 0$. $3x\,(x + 3) = 0$

28. $7n^2 = 10n$.

29. $2p^2 = 10p - 4p^2$.

30. $3r - 6r^2 = 0.$
31. $3x^2 + x - 7 = 8x + x^2 - 7.$
32. $x^2 = ax.$ solve for x.
33. $bx^2 = ax.$ solve for x.
34. $n^2 + an = -bn.$ solve for n.

35. If two rods be added to one side of a square field the area of the resulting field will be 48 square rods, what was the area of the original field?

This last problem seems to baffle us. It gives another sort of quadratic equation and our previous methods do not seem to work; the square root method might work if it were not for that bothersome 2x, and the factoring if it were not for the 48. But it may be that $x^2 + 2x - 48$ could be factored if we only knew how.

The difficulty to be overcome then lies in the little matter of factoring. We will therefore set ourselves to the task of overcoming that difficulty and learn how to factor some of the simpler expressions, considering in this chapter but four types. There are many more types, but four will be enuf for our present purposes. It might be well for you to realize that these are of very great importance and should be thoroly mastered.

25. We have just had instances of the simplest types of expressions that can be factored: expressions in which the same factor appears in each term.

We call this

Type 1. $ab + ac$. A common factor.

The type formula is $ab + ac = a\ (b + c)$.

Factor the following expressions:

1. $ax + ab.$
2. $4ac + 2a.$
3. $t^2 + 5t.$
4. $4a^2 - 6a^2b.$
5. $3x^2 - 6x.$
6. $3r - 6r^2.$
7. $10s^2n + 5sn^2.$
8. $3x^2 - 6x^3.$
9. $7y^2 - 6y.$
10. $7a^3 - 21ab^3.$

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11. $a^3bx - ab^2c$.
12. $4n^2 + 3n + 2$.
13. $4n^2 + 3n + 2n^3$.
14. $4x^3 - 2x^2 + 3x$.
15. $3x^2y + 2ab$.
16. $5nt - 7n^2t^2$.
17. $4x^3y - 12xz^2y^3 + 8y^3$.
18. $pt - t^2 + p^2$.
19. $pt + t^2 + p^2t + t^2p$.
20. $h^6 - ah^5 + bh^4 + ch^3$.
21. $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$.

26. $(n + 3)\ (n - 5)$ means, of course $(n + 3) \times (n - 5;)$ *i.e.,* $(n + 3)\ n - (n + 3)\ 5$, which of course equals $n^2 + 3n - 5n - 15$, which upon collecting terms, equals $n^2 - 2 - 15$.

You will notice that each term of one factor has been multiplied into every term of the other factor. It is often convenient to arrange the work just as long multiplication is arranged in arithmetic, except that here it is the custom to begin the multiplication with the left hand terms, because it is a little more convenient to do so.

$$\begin{array}{rrr}
n & -5 & \\
n & +3 & \\
\hline
n^2 & -5n & \\
 & +3n & -15 \\
\hline
n^2 & -2n & -15
\end{array}$$

Find $(x - 7)\ (x + 9)$.
Find $(2x - 3)\ (x - 5)$.
Find $(3x + 7)\ (2x - 9)$.

27. Perform the operations indicated and be sure to notice any peculiarity in the results.

$(x + 2)\ (x - 2)$, $(2n - 5)\ (2n + 5)$
$(a + 7)\ (a - 7)$, $(x - a)\ (x + a)$

Use what you have discovered in writing out at once the following expansions.

1. $(x + 11)\ (x - 11)$.
2. $(t + s)\ (t - s)$.
3. $(y - 16)\ (y + 16)$.
4. $(a + b)\ (a - b)$.
5. $(t + 7)\ (t - 9)$.
6. $(15a + b)\ (15a - b)$.

7. $(3x+5)\ (3x-5)$.
8. $(n^2-2)\ (n^2+2)$.
9. $(9-n)\ \ (9+n)$.
10. $(ab-2c)\ \ (ab+2c)$.

This developes

Type 2. a^2-b^2. The difference of two squares.

What are the characteristics of the form a^2-b^2 by which it can be distinguished from all other forms?

Thus the formula for Type 2 is $a^2-b^2=(a+b)(a-b)$.

State in words:

Factor:

11. x^2-16
12. b^2-a^2
13. s^2-36
14. n^2-9
15. y^2-25
16. $4x^2-25$
17. $4t^2+100$.
18. a^4-b^4.
19. $9-x^2$
20. $4x^2-100$.
21. $36x^2-49y^2$.
22. $64n^2-9m^2$.
23. $1-49n^2$.
24. a^4-25b^2.
25. $4n^2-m^2t^2$.
26. t^2-2p^2.
27. x^2-15.
28. x^3-16x^2.
29. $121-81x^2$.
30. t^2-p^2.
31. $(a-b)^2-c^2$.

28. Is there anything peculiar about the expansion of the following that will help you in writing out the results at once?

$$(x+2)\ \ (x+2),\quad (x+7)^2,\ (x+a)^2.$$

Use what you have noted in expanding the following:

1. $(x+5)^2$.
2. $(t+11)^2$.
3. $(y+1)^2$.
4. $(x+5)^2$.
5. $(h+a)^2$.
6. $(2a+3b)^2$
7. $(3n+5)^2$.
8. $(a+b)^2$.

This reveals a third type,

Type 3. $a^2+2ab+b^2$, a square trinomial.

What are the characteristics of this type?

The type formula is $a^2+2ab+b^2=(a+b)^2$.

State in words thus:—The square of the sum of two numbers is, etc.—

Factor:

9. $a^2 + 2ab + b^2$.
10. $x^2 + 6x + 9$.
11. $x^2 + 2x + 1$.
12. $n^2 + 8x + 16$.
13. $n^2 + 12n - 36$.
14. $n^2 + 12n + 36$.
15. $t^2 + 4t + 6$.
16. $t^2 + 4t + 4$.
17. $t^2 + 16t + 64$.
18. $x^2 + y^2 + 2xy$.
19. $4s^2 + 1 + 4s$.
20. $4x^2 + 12xy + 9y^2$
21. $(a - b)^2 = ?$ In what way does this differ from $(a + b)^2$.

Expand.

22. $(x - 7)^2$.
23. $y - 3t)^2$.
24. $(r - 11)^2$.
25. $(2a - 9)^2$.

Factor.

26. $a^2 - 2ac + c^2$.
27. $y^2 - 10y + 25$.
28. $n^2 - 16n + 64$.
29. $25x^2 + 4 - 20x$
30. $9x^2 + 25 - 30x$
31. $25a^2 - 20ab + 4b^2$.
32. Solve $x^2 - 2x + 1 = 0$.
33. Solve $4 + 121k^2 - 44k = 0$.
34. Solve $64t^2 + 4 = 32t$.

Now try the example that started us on this search. Can you do it. Why?

The two formulae under the type just considered may be written with a double sign.

$$a^2 \pm 2ab + b^2 = (a \pm b)^2.$$

29. Expand $(x + 7)\ (x + 5)$, $(x + 3)\ (x + 4)$.

Do you notice any peculiarity in the results that would enable you to write out the result at once? Apply to the following.

1. $(x + 3)\ (x + 9.)$.
2. $(n + 2)\ (n + 5)$.
3. $(x + 7)\ (x + 10)$.
4. $(y + 3)\ (y + 5)$.
5. $(a + 11)\ (a + 8)$.
6. $(x + a)\ (x + b)$.

This reveals still another type: The product of two binomials having a common term.

Type 4. $(x + a)\ (x + b) = x^2 + (a + b)\,x + ab$.

What are the characteristics of this type.

$$x^2 + (a + b)\,x + ab.$$

Factor:

7. $x^2 + 5x + 6$.
8. $n^2 + 7n + 5$.
9. $x^2 + 7x + 6$.
10. $t^2 + 23t + 102$.
11. $n^2 + 11n + 24$.
12. $a^2 + 13an + 12n^2$.
13. $h^2 + 11h + 30$.
14. $x^2 + 4xy + 3y^2$.

Will the same method of expansion apply to the case

$$(x - 3)\ (x - 7)?$$

15. $(x - 7)\ (x - 10)$.
16. $(n - 11)\ (n - 5)$.
17. $(x - 8)\ (x - 9)$.
18. $(x - 2y)\ (x - 3y)$.
19. $(t - 3)\ (t - 4)$.
20. $(x - a)\ (x - b)$.

Factor:

21. $n^2 - 7n + 6$.
22. $t^2 - 6t + 7$.
23. $s^2 - 4t + 3$.
24. $s^2 - 4s + 3$.
25. $b^2 - 29b + 190$.
26. $h^2 - 3h + 2$.
27. $x^2 - 23xy + 132y^2$.
28. $x^2 - 3x - 28$.

Will the same method of expansion apply in case

$$(x + 7)\ (x - 3)?$$

29. $(y + 8)\ (y - 7)$.
30. $(t - 1)\ (t + 2)$.
31. $(n - 10)\ (n + 3)$.
32. $(y + 3)\ (y - 25)$.
33. $(x + 6)\ (x - 7)$.
34. $(x + a)\ (x - b)$.

Factor:

35. $x^2 - 3x - 28$.
36. $t^2 + 6t - 7$.
37. $x^2 - 7y - 18$.
38. $p^2 + p - 132$.
39. $y^2 - 7y - 18$.
40. $r^2 - 11r - 60$.
41. $x^2 - 5xy - 84y^2$.
42. $y^2 + 7y - 60$.
43. $t^2 + 13t - 300$.
44. $z^2 - 9z - 36$.
45. $x^2 - 8x - 20$.
46. $x^2 - xy - 2y^2$.
47. Now try the field problem.

30. We have thus found four types of expressions which can be easily factored; others will be found later, but these are sufficient for our present needs. To repeat, they are:—

Type 1, $ab + ac = a(b + c)$.
Type 2, $a^2 - b^2 = (a + b)(a - b)$.
Type 3, $a^2 \pm 2ab + b^2 = (a \pm b)^2$.
Type 4, $x^2 + (a + b)x + ab = (x + a)(x + b)$.

A knowledge of these four types will give us power to factor a large number of expressions, tho we will find many that will resist our attempts and these must be left until our acquaintance with type forms is larger.

When you wish to factor an expression the way to proceed is clear.

1st. Find out whether the expression belongs to one of the types with which you are acquainted.

2d. If it does, it is readily factored.

3d. If it does not belong to one of those types you will have to discover some other type or wait until someone can show you what to do.

It is safe to say such expressions cannot be factored? Why?

PROBLEMS AND EXERCISES.

31 Factor the following expressions, stating the type or types of each.

1. $x^2 - 25$.
2. $a^2 + 2a - 80$.
3. $t^2 - t - 90$.
4. $b^2 + 16$.
5. $p^2 - 2p + 1$.
6. $p^2 + 2p - 32$.
7. $n^3 - n$.
8. $9x^2 - 12x$.
9. $n^2 + n + 1$.
10. $x^2 - 14x + 49$.
11. $4x^3 + 8x^3y + x^3y^2$.
12. $3x^4 - 3x^3 - 36x^2$.
13. If 2 rods be added to one side of a square field and 4

rods to the other side the area of the resulting field will be 48 square rods; what was the area of the original field?

Will both roots serve as answers to the problem? Why?

14. Find two numbers one of which is four times the other and whose product is 196.

15. Find a number such that five times its square increased by ten times itself equals 495.

State a rule solving quadratic equations.

16. Take series of integral numbers, say 5, 6, 7, 8, 9. Two numbers next to each other are called consecutive. If n be any number, what is the next consecutive number?

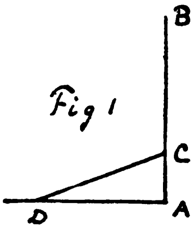
Write a series of five consecutive numbers wih t as one of them.

17. Find three consecutive numbers such that their sum is the equal of three-sevenths of the product of the last two.

18. The difference between the square of two consecutive numbers if 49, what are the numbers?

19. A walk containing 784 square feet is to be built around a garden 50 by 40 feet; how wide must the walk be? Draw a diagram before solving.

20. A flag-staff, AB, 50 feet high, was broken off at the point C: the broken end rested on C and the top touched the ground at D, thirty feet from the base of the staff. Find the length of the part still standing.



21. In solving the last problem you have used a certain law of the triangle. State it. Can you show that it is true? It was discovered by Pythagoras and is known as the Pythagorean Theorem.

22. $x^2 - x - 12 = 0.$

23. $n^2 - 4n = 45.$

24. $x^2 + 32 = 12x.$

25. $m^2 - 10m = 0.$

26. $7x^2 = 567.$

27. $2T^2 + 6T = 0.$

28. $3r^2 + 12r + 9 = 0.$

29. $p^2 + 9 = -6p.$

30. $11x^2 - 9 = 35.$

31. $h + 1/h = 2h.$

32. $(x - 3)(x + 2) = 19 - x.$

33. $x(x + a) = a(x + 1) + a(a - 1).$

34. $3/x - 1/x = \dfrac{5 + (x - 5)}{8}$

35. $n(n - 6) = 7n - 42.$

36. $(t + 1)^2 + (t + 2)^2 = 25.$

Factor:

37. $81y^2 - 9x^2.$

38. $3x^7 - 6x^5 + 3x^3.$

39. $h^2 - 8h + 16.$

40. $25s^2 - 36t^2.$

41. $k^2 - 7kt + 98t^2.$

42. $x^4 - y^4.$

43. $x^2 - x + 6.$

44. $(a + b)^2 - c^2.$

45. $N^2 - 34N + 288.$

46. $a^2 - (b + c)^2.$

47. $ca^2 + 3a^3b + 4a^4cb.$

48. $x^2 = t^2.$ solve for x.

49. $x^2 = a^2 + b^2 + 2ab.$ solve for x.

50. $x^2 - 3a^2 = ax.$ solve for x.

51. $\dfrac{n + 3}{4} = \dfrac{4}{n - 3}$

52. $\dfrac{x^2 - 9a}{4a} = 10.$

53. $\dfrac{x + 2}{x - 2} + \dfrac{x + 2}{x - 2} = 3.$

54. Find the surface area and the volume of the cylinder furnished you. Write a formula for each.

55. The surface of a cylindrical box is to be 261 square inches, the diameter is to be 7 inches: what will be the height?

56. The surface of a cylindrical box is to be 251 square inches, the height is to be 6 inches, what will be the radius? The diameter?

What do you think had best be done with this last problem?

32. Find the value of $75^2 - 25^2$.
$323^2 - 277^2$.

Now find he required values in another way:—by factoring the expressions first and then computing.

Factoring is of use for other purposes than solving equations. A great many arithmetical computations can be greatly simplified by taking care to factor as much as possible before performing the calculations.

57. It is desired to find the distance across a lake. A and B are stakes on opposite sides of the lake. BC at right angles to BA is measured and found to be 375 feet, AC is found to be 575 feet.

What is the distance BA?

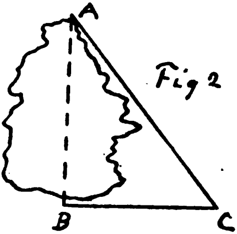


Fig 2

58. What will be the cost of laying a four-foot concrete walk around a circular tower 20 feet in diameter, at 90 cents a square yard?

59. Draw two circles with the same center, and with radii 2½ inches and 2 inches. What is the area between them?

60. Draw two circles with the same radii as in the last example but so that the circles touch each other on the inside of the larger one. Find the area of the crescent shaped figure.

61. What will it cost to cement the walls and floor of a cylindrical silo 30 feet high and 15 feet in diameter, at 25 cents a square yard?

62. Find value of $1639^2 - 739^2$.

63. Find value of $16.1(4.31^2 - 2.31^2)$.

64. Find the value of $r^2 - 5r + 6$, when $r = 3.71$.

CHAPTER 3.

GRAPHICS.

33. Draw a straight line on a sheet of paper and mark a point about the middle of the line 0.

Now mark a point on the line two inches from 0.

How many such points can you determine? How shall we designate them so as to distinguish the one from the other?

Thus one can determine the position of any point on a line by measuring its distance from a fixed point, 0, taken as the starting point or origin as it is called.

1. Lay off and tag properly the points 3, 1.5,—1/2, 0,—0,3/4. 1/4, .7, 1.25, —2, —3, and have your paper ready to hand in.

2. Determine and mark the distance of each point from 0, on the paper handed you and hand in the paper when you have finished.

34. 3. Take a sheet of white paper, make two dots on it anywhere. Can you describe those two dots so that I can put dots in a similar position on my sheet of paper?

How is the position of a place on the map determined? How is the position of a place on the surface of the earth designated?

4. Take another piece of paper and plot the following points and hand in your paper. r means to right, u means up.

r 2,	r 4,	r 1,	r 5,	r 0,	r 3,	r 0.
u 3,	u 2,	u 3,	u 5,	u 2,	u 0,	u 0.

5. The two numbers associated with each of the points ar called the co-ordinates of the point.

Determine the co-ordinates of the points on the paper handed you and have your paper ready to hand in.

6. Upon another sheet of paper draw two lines perpendicular to each other. How will you draw the lines?

Call the up and down line the vertical axis, and the other line the horizontal axis.

Now place a point 2 inches from the vertical axis and 3 inches from the horizontal axis. How many such points can you place? Can you devise any way to distinguish between these points?

7. On another sheet plot the following points and hand in your papers. The first number in each pair is the distance of the point from the vertical axis, and the second number the distance from the horizontal axis.

$(3, 4)$, $(-2, 3)$, $(-1, -2)$, $(2, -4)$, $(0, 1)$, $(0, 0)$.

8. Determine the co-ordinates of the points on the paper handed you. Mark them as above (a, b) and hand in.

DEFINITIONS.

35. The two lines are called the axis,—the vertical axis and the horizontal axis. The point where the two axis intersect is called the origin.

The numbers designating the position of a point are called the co-ordinates of that point.

Co-ordinates measured to the right are +.
Co-ordinates measured to the left are —.
Co-ordinates measured up are +.
Co-ordinates measured down are —. Why?

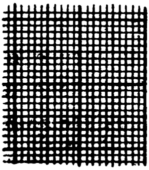


Fig 3

The point is called the graph of the co-ordinates.

The plotting of points is made much easier if co-ordinate paper is used. In the illustration the lines are drawn twenty to the inch.

Google

36. 9. Using two of the darker lines near the middle of the page of co-ordinate paper as axes, and letting each of the smaller divisions denote unity, plot the points: (10, 20) (15, 15) (0, 40), (20, —10) (— 5, —45) (— 30, 0) (35, 60) (0, 50).

10. We will designate north latitude by + and south latitude by —, east longitude by +, and west longitude by —. Locate on paper the following places:

	LATITUDE	LONGITUDE
London	51½°	0
Paris	49	2½
Berlin	52½	13½
Dublin	53⅓	— 6½
Madrid	40½	— 3½
Gibralter	36	— 5½
The Hague	52	4½
Rome	42	12½
Edinburgh	56	— 3

11. The hourly temperature readings at Peoria for July 17, 1904, the hottest day of the year, were:

6 A. M.	77°
7 A. M.	78
8 A. M.	80
9 A. M.	83
10 A.M.	85
11 A. M.	88
12 N.	89
1 P. M	90
2 P. M.	92
3 P. M.	93
4 P. M.	93°
5 P. M.	92
6 P. M.	92
7 P. M.	90
8 P. M.	89
9 P. M.	87
10 P. M.	85
11 P. M.	84
12 P. M.	82

Plot these numbers. Take the lower left hand corner of sheet as origin, lay off the hours horizontally using ¼-inch to each hour beginning at 12 midnight. Lay off temperature scale vertically, using $\frac{1}{10}$ of inch to degree.

The points thus plotted form a sort of regular figure. If we had the temperature readings for every quarter hour and should plot them, we would find that the dots would be much closer together and would seem to be on a curve. This temperature curve shows the changes in temperature much more clearly than does the table of readings.

12. Draw the temperature curve for Peoria, June 29, 1906.

HOURLY TEMPERATURE READING

12 midnight	77	11 a.m.	88
1 a.m.	75	12 noon	91
2 a.m.	74	1 p.m.	92
3 a.m.	73	2 p.m.	92
4 a.m.	73	3 p.m.	83
5 a.m.	72	4 p.m.	68
6 a. m.	75	5 p.m.	70
7 a.m.	76	6 p.m.	76
8 a.m.	81	7 p.m.	76
9 a.m.	84	8 p.m.	76
10 a.m.	81	9 p.m.	76

Note the sudden drop in temperature between 2 and 5 o'clock. There was a severe storm at that time.

13. Contrast the curve of the 11th example for the hottest day of 1904 with the curve for the coldest day of the following winter, the readings for Dec. 26 are given below:—

Temperature readings for Dec., 25, 26, 1903.

Dec. 25,	6 a.m.	18°		12 a.m.	-5
	8 a.m.	18	Dec. 26,	6 a.m.	-7
	10 a.m.	21		8 a.m.	-6
	12 a.m.	28		10 a.m.	0
	2 p.m.	12		12 a.m.	5
	4 p.m.	8		2 p.m.	12
	6 p.m.	4		4 p.m.	14
	8 p.m.	0		6 p.m.	16
	10 p.m.	-2		8 p.m.	18

The only points on these temperature curves we have just drawn that we know, are the hourly points.

Over at the weather bureau there is an instrument called a thermograph which draws the curve automatically, thus giving the actual temperature for every moment of the day.

From such a curve the temperature at any time of the day or night may be read off.

14. Here is a temperature curve from the Weather Bureau. Read off the temperature at 6 a.m., at 9 a.m., at 10:30 a.m., at 12 n., at 3 p.m. and at midnight.

15. Plot the population curve of the United States for the years 1800-1900. Lay off dates horizontally, plot to the nearest half-million and choose the scale so that you can get the graph on one sheet.

YEAR	POPULATION	YEAR	POPULATION
1800	5,008,483	1860	31,443,321
1810	7,229,881	1870	38,553,371
1820	9,633,822	1880	50,155,783
1830	12,866,020	1890	62,622,250
1840	17,069,453	1900	76,304,799
1850	23,191,876		

16. Below is a table giving the monthly mean rainfall in inches for Peoria from 1855 to 1885, and for the valley of the Chagres river above Bohio where a large dam creates a lake which is to be used as a part of the Panama canal. Plot on the same sheet.

MONTH.	PEORIA. INCHES.	CHARGRES VALLEY INCHES.
January	1.61	5.78
February	2.04	1.06
March	2.68	1.34
April	3.17	2.95
May	3.80	10.47
June	4.03	10.63
July	4.04	15.51
August	3.16	14.21
September	3.49	11.45
October	2.71	14.21
November	2.37	17.01
December	2.48	6.92

17. The table below gives the soundings of the bottom of the river for every hundred feet straight across the river from the foot of Main Street. Plot the profile of the bottom of the river at that point.

0.0
8.1
10.6
10.3
10.1
10.3
10.5
10.1
9.4
9.5
9.2
8.2
7.4
6.5
5.3
3.9
2.8
1.4
0.8
0.0

18. Here are similar surroundings across from Kelly's Beach, those marked — are at Towhead Island:

0.0
3.6
5.5
8.6
20.1
18.6
17.3
15.1
4.6
3.6
3.6
4.5
4.6
4.6
4.6
5.7
5.6
7.1
7.9
7.4
8.6
8.1
5.1
3.4
1.8
0.0
— .4
—1.4
—2.4
—2.4
—1.4
— .4
0.0
0.0
.2
.4
1.0
1.7
2.9
2.6

37. 19. While riding in an automobile on a level street, where the blocks were 12 to the mile, I noted and tabulated as follows, the times at which we crossed the street inter-sections.

BLOCKS	TIME	BLOCKS	TIME
0	10:17	11	10:22.5
1	10:17.5	12	10:23
2	10:18	13	10:23.5
3	10:18.5	14	10:24
4	10:19	15	10:24.5
5	10:19.5	16	10:25
6	10:20	17	10:25.5
7	10:20.5	18	10:26
8	10:21	19	10:26.5
9	10:21.5	20	10:27
10	10:22		

Draw the time space curve, use scale ¼-inch to block, ½-inch to minute.

These plotted points lie on what sort of a line? None of the other graphs you have drawn have been like this one. This case differs in a very important respect from all the others we have considered. Let us discover in what way the data of this case differs from the data of the 14th?

TIME	INCHES	SPACE	TIME
0	5.78	0	0
1	1.06	1	½
2	1.34	2	1
3	2.95	3	1½
4	10.47	4	2
5	10.47	5	2½
6	15.51	6	3
8	11.45	8	4
10	17.01	10	5

It is evident that in the automobile case the numbers in the cond column are twice the corresponding number in the first column that is, the space is twice the time, which is a law that can be expressed in the equation

$$s = 2t;$$

Whereas in the case of the rainfall we can detect no such law connecting the numbers.

In the equation $s = 2t$, which shows the relation between the co-ordinates of any point on the graph drawn, s stands for space in blocks; t stands for time in minutes. What does 2 stand for? Express the rate in miles per hour.

In this way we have two ways of expressing the same law—the law of uniform motion; the graph and the equation.

The co-ordinates of every point on the line just drawn will satisfy the equation $s = 2t$, and we can read off the corresponding numbers from the line just as in the case of the temperature curve. How long will it take the automobile to go 3½ blocks? How far will it go in 2¼ minutes?

These results can be computed from the equation; they can be read from the graph a little more quickly.

The graphs of any set of pairs of numbers can be drawn. While in many cases no law can be found connecting the numbers, in other cases such a definite law can be stated.

20. Plot the natural numbers horizontally and their corresponding squares vertically. Let the horizontal scale be unity, ¼-inch and the vertical scale be unity, $\frac{1}{20}$-inch. Draw a smooth line thru the points located and also state the equation.

Read off from the graph the square of 5½, 7.2, the square root of 110, 130, 28, 61.

21. The sum of two numbers is 8. What are the numbers? Calling one number x and the other y, make a list of all numbers whose sum is 8. Plot these numbers using the same unit on each axis. Let the horizontal axis be the x-axis. State the equation between x and y.

22. Make a table of numbers whose product is 12. Plot them and write the equation.

23. Make a table of numbers whose quotient is 12. Plot them and write the equation.

24. The sum of two numbers is 12. First write the equation. Then compute a series of values satisfying the equation by assigning x consecutive values: -3, -2, —, 0, 1, 2, 3, 4, and computing the corresponding values for y. Plot these sets of numbers. The line is the graph of the equation.

25. State the equation showing the total cost of n yards of muslin at 15 cents a yard and draw the cost-number graph.

26. State the interest equation for $1.00 at 6% for t years and draw the interest-time graph. Use scale; vertical, one inch to the year; horizontal, half inch to 10 cents.

Show how this graph may be used in finding the interest for any number of years.

27. Draw the graph of $p = 3t + 4$.

28. Draw the graph of $2s + 3k = 12$.

29. A spiral spring is hung in front of a meter stick. Read and record the length of the spring. Place a steel ball (bicycle ball) in the pan attachd to the spring and read and record the length of the stretched spring:

Repeat with 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 balls, and tabulate readings thus:—

NO. OF BALLS	OBSERVED READING IN mm	LENGTH STRETCHED IN mm
0	—	—
1	—	—

Now plot the numbers found in the 1st and 3d columns, laying off numbers of balls horizontally and the length stretched vertically. What does the graph seem to be? State if you can an equation for this graph. To test the accuracy of your guess find several pairs of numbers which will satisfy the equation and then plot these numbers on the figure. What are your conclusions?

30. Balance a meter stick at its center of gravity. At (d) 20 cm from the center hang a 50 gram weight (w), on the other arm of the balance slide a 20 gram weight (w^1) to (d^1) until the balance is restored.

Repeat with 40, 50, 70, 100, 200, 300 gram weights in place of (w^1). Record results in tabulated form.

Results of Experiment.

d	w	w × d	d^1	w^1	$w^1 \times d^1$
20	50	1000	—	20	—
			—	—	—

Fill in the column 3, and write an equation connecting the numbers d, w, d^1, w^1.

From the equation determine where to place a 150 gram weight in order to restore the balance, and verify the result by experiment: try the same for 60 grams.

What weight must be placed at the distances 25 cm., 5 cm. in order that the balance be restored. Verify by experiment.

Plot the equation $wd = 1000$.

Plot the numbers found in the 4th and 5th columns.

In the following examples take the horizontal axis as the x-axis and use the same scale on both axes. Draw the graphs to the scale indicated assigning the values: —5, —4, —2, —1, 0, 1, 2, 3, 4, 5 to x and computing y.

31.	$y = 4x - 2$.	Scale unit ½ inch.
32.	$2x + 3y = 12$.	Scale unit ½ inch.
33.	$2y + x + 4 = 0$.	Scale unit ½ inch.
34.	$4y = x^2$.	Scale unit ½ inch.
35.	$xy = 12$.	Scale unit ½ inch.
36.	$20y = x^2$.	Scale unit ¼ inch.
37.	$5y = x^2 - 4x$.	Scale unit ½ inch.
38.	$4y = 16 - x^2$.	Scale unit ½ inch.

39. Two trains moving in the same direction at the rate of 30 miles and 50 miles an hour respectively leave M at 9 and 10

o'clock. They meet at N. What is the distance between M and N.? At what time do they arrive at N.?

Solve this in two ways, by means of equation and by means of graphs.

A train leaves P, 100 miles from N, at 9 o'clock, at what rate must it move in order to meet the other train at N.?

40. P is a town 90 miles west of Q. At eight o'clock an east bound train leaves P running at the average rate of 20 miles an hour; a west bound train leaves Q, running at the rate of 35 miles an hour. Where and when will they meet?

41. The limited express bound for the east leaves P at 8 o'clock running at the average rate of 60 miles an hour. A west bound freight leaves Q at the same time. What must be its average running rate in order to meet the express at a siding 24 miles west of Q.

42. A and B are two towns 110 and 80 miles west of C. At one o'clock an east bound passenger, average running rate 40 miles an hour, leaves A. An hour and a half later the west bound limited leaves C at an average running rate of 60 miles an hour. What must be the running rate of a freight which left B at one o'clock so that it may meet the two passengers at the same place and time?

CHAPTER 4.

LINEAR EQUATIONS INVOLVING TWO UNKNOWN NUMBERS.

38. Consider the problem:—If the sum of two numbers is 8, what are the numbers?

Stated in algebraic language the problem would read:

If $x+y=8$, find x and y. (1).

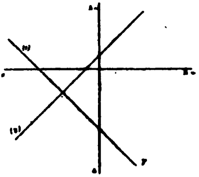
How many answers can you give to the problem?

Each set of values that will satisfy the equation is called a solution: There are two numbers in every solution.

The graph of the equation is the straight line l in the figure.

There is an unlimited number of points on the line. The pair of co-ordinates belonging to each point is a solution of the equation.

There are then an unlimited number of answers to the problem given.



Show in the same way that there is an unlimited number of answers to the problem: If the difference of two numbers is 2, what are the numbers?

i. e. $x-y=2$ (2).

Neither of these problems taken alone has a definite solution.

But suppose we combine them into one. If the sum of the two numbers is 8, and their difference is 2, what are the numbers?

$$x+y=8 \qquad (1)$$
$$x-y=2 \qquad (2)$$

Let us draw the graphs on the same figure.

It is clear that the co-ordinates of the point common to the two lines will satisfy both equations, and will thus furnish a definit answer to the problem. If the graphs have been carefully drawn, these co-ordinates may be read off the figure, and the problem is solved. To check the work, the numbers of the solution should be substituted in each equation. Work out the problem just given.

39 An equation such as, $x+y=8$, of the first degree in each of the unknowns is called a linear equation because its graph is a straight line. It will be proved to you sometime that the graph of every equation of the first degree in two unknowns is a straight line.

In drawing a graph we have had to find several sets of values satisfying the equation and then plot them, which is rather tedious. Now that we know that the graph of a linear equation is a straight line we can save ourselves much work.

How many straight lines can you draw thru one point?

How many straight lines can you draw thru two points?

How many points does it take to determine the position of a straight line?

How many sets of values do we need in order to draw the graph of a linear equation?

Very good! but we can do still better.

Does it make any difference what sets we use?

Let us choose the sets which can be found with the least trouble.

What would you advise?

The points where the graph cuts the axis are the best to choose. Why?

For $x+y=8$ $\quad$ $x=0,\ y=8.$

$y=0,\ x=8.$

Solve by the graphical method:

1. $x+2y=1.$
   $x-2y=5.$

2. $x+y=1.$
   $x-y=1.$

3. $2s+3t=19.$
   $3s+2t=16.$

4. $v-2t=2.$
   $2t-6v=3.$

5. $12n-5m=3.$
   $3n+4m=3.$

40. The results obtained from the last example indicate that the graphic method is not as accurate as is to be desired, depending as it does upon the accuracy of the measurements. A method is needed that will give better results.

Take again the first problem of the chapter.

(1) $x+y=8,\ x-y=2.$ (2)

If we could get rid of one of the letters all would be easy. Let us go about it in this way:

Solve (1) for x $x=8-y$

Substitute for x in (2) $(8-y)-y=2$ (3)

This is a simple equation.

Remove parenthesis and unite terms $8-2y=\cancel{6}\ 2$

Transpose $2y=6$

$y=3$

Then since $x=8-y$

upon substituting 3 for y $x=8-3$

$=5$

and the solution is $x=5,\ y=3$

In getting the equation (3) we literally turned x out of doors. Mathematicians usually say "we eliminated the x" The word eliminate is from a latin verb *elimino,* which means to turn out of doors.

The method just given is called the substitution method.

State a rule for solving a set of equations in this way.

Solve by the substitution method:

6. $a=9b.$ $4a-7b=5.$

7. $p-2r=8r-1.$ $2p-4r=p+r+9.$

8. $2x+y=7.$ $3y-2x=13.$

9. $x+y=2y+7.$ $3y=5(x+1).$

10. $a:b=3:4.$ $a+b=14.$

11. $2n-3p=12.$ $3n+5p=8.$

12. $11x+8y=76.$ $11x+7y=72.$

41. The last two examples were a little more troublesome than some of the others. Let us follow the usual course in Mathematics and look for an easier way. Try the 12th again thus:—

$$11x + 8y = 76. \qquad (1)$$
$$11x + 7y = 72. \qquad (2)$$

We know that if equal numbers be subtracted from equal numbers the result will be equal.

Let us subtract the members of (2) from the corresponding members of (1) and get

$$y = 4.$$

We certainly have gotten rid of x in a hurry.

Substitute this in (1) and get

$$\begin{aligned} 11x + 32 &= 76. \\ 11x &= 44. \\ x &= 4. \end{aligned}$$

and the solution is $x = 4, y = 4.$

Try the same method on

$$5t + s = 6. \qquad (1)$$
$$4t - 3s = -5. \qquad (2)$$

Add member to member, $9t - 2s = 1.$

This does not seem to help any. If the coefficients of the s were the same, it would be different. They will be the same if we multiply (1) by 3, and get $15t + 3s = 18.$

$$\begin{aligned} 4t - 3s &= -5. \\ 19t \qquad &= 13. \\ t &= \frac{13.}{19} \end{aligned}$$

and substituting in (1)

$$5.\frac{13}{19} + s = 6.$$

$$s = \frac{49}{19} = 2\frac{11}{19}$$

Let us try the same method on

$$2n - 3p = 12. \qquad (1)$$
$$3n + 5p = 8. \qquad (2)$$

Let us eliminate p. To make the co-efficients of p the same in the two equations, multiply (1) by 5 and (2) by 3.

$$10n - 15p = 60. \qquad (3)$$
$$9n + 15p = 24. \qquad (4)$$

adding the number of (3) and (14) will eliminate the p.

$$19n = 84.$$
$$n = \frac{84}{19} = 4\frac{8}{19}$$

substitute in (2) and get

$$3 \cdot \frac{84}{19} + 5p = 8$$
$$p = -\frac{20}{19} = -1\frac{1}{19}$$

This is called the addition method.

State a rule that will cover the case.

Apply this method to

13. $17x - y = 31$
$15x + 3y = 39.$

14. $2x + 7y = 38$
$3x + 4y = 31.$

15. $5 + p + 2y = 0$
$7 + 5p + y = 0.$

16. $17a - 18b = 15$
$5a + 12b = 39.$

17. $3v + 4w = 3$
$12v - 5w = 3.$

18. $v - u = a$
$v + u = e.$

Find u and v.

42. In solving the following use either of the last two methods, choosing the one you think the best for that particular example.

19. $4x + 3y = 6{,}000$
$x + 2y = 2{,}000.$

20. $3A = 6$
$5A + B = 5.$

21. $24 = u + \frac{21}{2}f$
$32 = u + \frac{29}{2}f$

22. $4x - \frac{y}{5} = 6$
$\frac{x}{4} + \frac{y}{3} = \frac{23}{6}$

23. The sum of two numbers is n and their difference is m, what are the numbers?

24. Apply the formulae resulting from problem 20 to the special case; sum is 3675 and the difference 2691.

25. The sum of two numbers is 27, twice the first added to three times the second is 96, what are the numbers?

26. The difference between two numbers is 28, five times the first less the second is 197, what are the numbers?

27. These two pieces, one of wood and the other of metal are to be weighed: but we have nothing to weigh them with save these gram weights and the platform scales, and it is impossible to balance the scales with the weights at hand.

Let us, however, place the piece of wood (w) and the piece of metal (w^1) at different points on one arm of a meter stick hung at its center of gravity and the balance with a weight on the other arm (G).

Write an equation which will state the case. Now interchange the two unknowns (w) and (w^1) and write an equation which will state the second case. Now find the required weights.

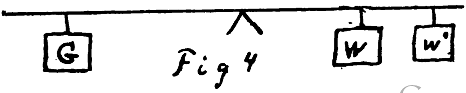


Fig 4

28. What is the fraction which is equivalent to $\frac{1}{3}$ when 1 is added to its numerator, but is equivalent to $\frac{1}{4}$ if 1 be added to its denominator.

29. Generalize the last problem and solve.

30. In the following example the known numbers are known approximately to decimal place; which of the three methods is the best in this case. Remember that the values of x and y are required to only one decimal place.

$$3.5x - 2.7y = 7.6$$
$$.8x + 1.2y = 4.1$$

31. Solve $\frac{y}{x} = 1$ $\qquad \frac{y}{x+100} = .5773.$

43. In this chapter we have learned that some problems may be solved by the use of two unknown numbers. We have seen that in such cases it is necessary to have two equations involving the unknowns.

Each equation of the sets we have considered was a linear equation.

We have learned three methods for solving such sets of equations.

equations. Describe each method in detail. Show the advantage of each.

What kind of a line is the graph of an equation of the first degree?

What is the simplest way of drawing the graph of a linear equation?

What is a linear equation?

What does elimination mean?

How may a letter be eliminated from an equation?

CHAPTER 5.

THE FUNDAMENTAL OPERATIONS.

44. Let us take account of the most important things we have learned from the last four chapters. We have seen that the language of algebra enables us to state certain facts we know about numbers in a short and clear way. It has given us an orderly manner of setting down the solution of many arithmetical problems.

We have discovered that the equation is a wonderfully useful help in solving problems, giving us the power to solve many a problem that we could not solve before. To show the truth of this statement and to illustrate how far we have advanced, try this problem by arithmetic.

Take a number, subtract 5 from it, multiply this result by 3, add the number itself and then subtract 3. The result is twice the original number. What was the original number?

Would any number do for a start? $3(x-5)+x-3$

Answer these questions by the aid of the equation. $3x-15+x$

Of course the problem just considered is not of much importance. It is more of a curiosity than anything else but it shows how strong we have grown to be.

We have learned how to handle three different kinds of equations so as to find the value of the unknown number; simple equations, quadratic equations, and a set of linear equations.

We found it a fundamental law of the equation that so long as each number of an equation is treated in the same way, the results will still be equal. It was in the light of this law that we were able to solve equations.

We found that the operations used in algebra were the same as those we were accustomed to use in arithmetic. We have seen that whereas in arithmetic it was always possible to actually perform the operation required, in algebra the operation can often be merely indicated.

We have added to our system of numbers that very important class of numbers, negative numbers, which have opened up a wide field for us to work in.

We have so far had comparatively little difficulty in dealing with general numbers (letters) equations and negative numbers, at least when once we have learned how.

We have caught a glimpse of factoring and have learned how to factor four types of expressions.

We also have been able to make a graphic picture of an equation which shows to the eye the relations of the number involved.

We have in this first taste of algebra, come in contact with the simpler problems. The operations required have been performed on the simplest of forms. But we will find as we take up more difficult problems that many very complicated expressions will arise which we must be able to untangle. And many of them are pretty puzzling. But there is usually a way out. We will now start on our way of conquest by undertaking a more careful and systematic consideration of the operations used in algebra.

The operations will be considered in the following order: addition, subtraction, multiplication, division, involution or finding a power, evolution or finding a root.

45. The definitions of this article are already known to you. They are given here that they may be recalled to your mind.

An algebraic expression is composed of arabic figures, letters and signs of operation and relation.

Arabic figures represent numbers of fixed value.

Letters represent general numbers.

Addition and subtraction are indicated by the sign + and —.

Multiplication is usually denoted by the omission of a sign : ab.

Division by the horizontal line $\frac{a}{b}$

If several equal factors are to be multiplied together such as a.a.a.a.a, it has been the practice for the last three hundred years to use the notation a^5 where the exponent means the number of factors.

We owe this exponential notation to a Frenchman, Simon Stevin, who lived at Bruges, Belguim, 1548—1620.

1. Addition.

46. When an algebraic expression is to be added or subtracted it is set off by placing a + or — sign in front of it. Such an expression is called a term.

$$3a + 2x - 3x^2$$

$3a$, $2x$, $3x^2$ are terms.

In case the expression to be added or subtracted is itself composed of several terms, it must be inclosed in parenthesis and the proper sign placed in front:

$$3a + (5b - 2c)$$

$5b - 2c$ is here considered a term.

The various parts of a term being multiplied together are called the factors of the term.

We often speak of a term as having two parts, a numerical part, and the literal part.

The numerical part is called the numerical co-efficient. Any one or more of the factors of a term is the co-efficient of the rest of the term.

$$4ax^2$$

4 is the numerical part.
ax^2 is the literal part.
4 is the numerical co-efficient.
a is the co-efficient of $4x^2$.
4a is the co-efficient of x^2.

Terms in which the literal parts are the same are called like terms. (Notice the exponents belong to the literal part of the term).

An algebraic expression of one term is called a monomial: one of the two or more terms is called polynominal.

There are special names to designate polynomials of different numbers of terms.

A binomial has two terms.

A trinomial has three terms, etc.

49. Addition is subject to two fundamental laws.

1. The law of order, or Commutative Law:

$$2 + 5 = 5 + 2,$$

or in general

$$a + b = b + a.$$

2. The Law of Association, or the Associative Law

$$2 + 3 + 7 = 5 + 7 = 2 + 10 = 12.$$

or in general

$$a + b + c = (a + b) + c = a + (b + c).$$

48. If several algebraic expressions are to be added or subtracted, it suffices to write them down in a horizontal row with the proper signs before each.

Such a polynomial is in a simpler form when all the like terms have been united.

Like terms can be united into one term by addition by adding the numerical co-efficients and prefixing that sum to the literal part of the term.

The sum of unlike terms can be indicated only:

$$3x^2y + 5x^2y - 2x^2y = 6x^2y.$$

$3x^2y + 2xy^2 + 5xy$. The terms are unlike

It is always well to arrange the terms of algebraic expressions in some regular order, such as in ascending or descending powers of some letter.

Simplify the following polynomials:

1. $4m + 3n - 2n + 5m - 7n - 3m$.

2. $3x + 4y - 2z - 7x + 9y + 11z - 13y - 10z$.

3. $3x^2 + 5x - 2x^2 + 2 - 3x - 5x^2 - 5$.

4. $3ab + 6a^2b + 7ab^2 - 2ab - 3ab^2 + 3a^2b$.

5. $3x^2y^2 + 5x^2y - 3xy^2 - 3x^2y + 2xy^2 - 6x^2y^2$.

6. $7a^2 + 7/3\ a + 9/5\ a^2 + 3 - 3/7\ a + 2a^2$.

7. $k^3 - 4k^2n + 6kn^2 + 2k^2n - 3kn^2 - 2n^3 + n^3 + 3k^2n + 4kn^2$

8. Find the sum of $3a^2$, $4b^2$, $-a^2$, $-5b^2$, $-2ab$, $5a^2$, $3ab$.

9. Find the sum of $a^2 - b$, $b^2 - a$, $2a^2 - 3b,^2$, $a + b$, $a^2 - a$, $b - b^2$.

It is often convenient to place the polynomials to be added under each other, like terms under each other and then add up the columns as in arithmetic.

Add the polynomials:

10. $a^3 + 3a^2b + 3ab^2 + b^3$, $a^3 - 3a^2b + 3ab^2 - b^3$, $-a^3 + 3a^2b - 3ab^2 + b^3$, $-a^3 - a^2b - 3ab^2 - b^3$, $a^3 + 6a^2b + 6ab^2 + b^3$.

11. $x^2 + x^3 + x^4$, $2x^3 - 2x - 2x^3$, $3x - 3x^2 + 3$, $2/3xy - 5/4y^2$.

12. $5/8\ x^2 - 1/3\ xy - 1/4\ y^2$, $-\cancel{x^2} - xy + 2y^2$, $3/2\ x^2 - 2/3xy - 5/4y^2$.

13. $1/2\ n^3 - 1/3\ n + 1/15$, $-1/3 + n^3 - 1/3\ n$, $2/15 + 2/3n - 3/2\ n^7$.

14. $ax + bx$ is how many times x?

15. $nxy^2 + mxy^2 + pxy^2$ is how many times xy^2?

In the following collect the terms in x^2, those in x, and the absolute terms and arrange in the descending powers of x. Can you put the expression in another form?

16. $ax^2 + bx + bx^2 + a + ax + b$.

✓17. $ax^2 + bx + c$, $bx^2 + cx + a$, $cx^2 + ax + b$.

18. Unite the terms in t, and t^2

$$3t + at - 3t^2 - bt^2 - bt + 5t - 7t^2.$$

19. $es + et + ev + ep$.

2. Subtraction.

49. Subtraction is the inverse of addition.

We have learned that the subtraction of a negative number gives the same result as the addition of a positive number of the same absolute value. Stated in algebraic language this is

$$-(-b) = +b.$$
$$-(+b) = -b.$$

Thus it is seen that a subtraction may always be replaced by an addition of the same number with its sign changed.

$$(3x - 9x^2) - (2x - 5x^2) = 3x - 9x^2 - 2x + 5x^2 = x - 4x^2.$$

Subtraction thus becomes very simple. Merely change the sign of each term of the expression to be subtracted and add.

Note that the parenthesis is merely a devise to denote that certain terms are to be considered as a whole.

Perform the following subtractions:

✓ 1. $(x^3 - x^2 + x + 1) - (x^3 + x^2 - x + 1)$. ✓

2. $(-3a^3 + 5b^3 - c^3) - (a^3 + b^3 - c^3 - 3abc)$.

3. $(2uv - v^2 - 3u^2) - (u^2 + v^2 - 2uv)$.

4. $3xy + 2x^2y - (4x^3 - 2xy^2 + 3yx)$.

5. $3x - 2(2x^2 - 3y) - 3(x + 2x^2 - y)$.

6. $x^2 - (x + 5)(x - 1)$.

7. $h^3 - (h + 1)^2 + 2(h - 2) - 5$.

8. $(n - 2) - 2(n - 2) - (3n - 6) + 5$.

.9 $\frac{4}{3}R^3 - (\frac{2}{3}R^3 - aR^2 + \frac{1}{3}a^3) - (\frac{2}{3}R^3 - bR^2 + \frac{1}{3}b^3)$

10. From $4n^2 - 2n + 3$ take $2n^2 - 5n - 7$.

This may be done in the same way as those above or we may do as we often do in arithmetic, place like terms under like terms and then subtract:—

$$\begin{array}{r} 4n^2 - 2n + 3 \\ 2n^2 - 5n - 7 \\ \hline 2n^2 + 3n + 10 \end{array}$$

11. From $-7x^2y^2 + 13x^3y + 15x^4y$
take $-4x^2y^2 \mp 7x^3y - 8x^4y$

12. From $5k^4 - 3k^3 - 2k^2 + 5k + 2$
take $2k^4 \pm 5k^3 \mp 2k^2 \pm 3k \pm 5$

13. From $72x^4 - 78x^3y - 10x^2y^2 + 18xy^3 + 3y^4$
take $\pm x^4 \pm 36x^3y \mp 17xy^3 \pm 34y^4 \pm 10x^2y^2$

14. Subtract $3x^2 - 7x + 1$ from $2x^2 - 5x - 3$, subtract that difference from zero and add this result to $2x^2 - 5x - 4$.

Simplify:

15. $5 - (y + (3 - 5))$.

16. $a - (b + (c - d))$.

17. $a - (b - (c + d))$.

50. We have seen that in simplifying expressions containing parentheses it does not do just to rub them out. It is all simple enough when one remembers the meaning.

Note that when a + sign stands before the parenthesis it may be removed without any other alterations; but when a — sign stands before, the signs of all terms within the parenthesis are to be changed and this expression added.

Skillful handling of parenthesis is of much importance.

18. $a - (b - (c - (d - e)))$.

19. $x^2 - (x - (2 + x^2) + (3 + x))$

20. $3x - 2(7x - (2x - 3))$

21. $10m^2 + (5mn - 6m^2 + n^2 - (2mn - (m^2 + n^2))) - n^2$

22. $3((n - (n - 3) - 2) + (2x - \overline{3 - 2x}))$

23. Place the last two terms of each expression in a parenthesis.

$$3x^2 + 2x + 4$$
$$x^2 - 2x + 4$$

24. Put the last three terms in parenthesis.

$$c^2 + a^2 - 2ab + b^2$$
$$c^2 - a^2 + 2ab - b^2$$

25. Put the terms in parentheses two by two
1st, beginning with the first term.
2nd, beginning with the second term.

$$a - b + c - d - e - f + g - h$$

26. Put the last two terms in a parenthesis.

$$x^3 - 4x^2 - 3x + 12$$

27. Put the last two terms in a parenthesis.

$$x^3 + 2x^2 - x - 2$$

Put in parenthesis by pairs.

28. $a^3 - b^3 - a^2 - b^2 + a - b.$

29. $-n + 1 + n^2 - 1 - n^3 + 1.$

30. $x - a - x^2 + ax + ax - a^2.$

3. Multiplication.

51. Multiplication is subject to certain laws.

1. The Law of Order, called the Commuative Law

$3 \times 5 = 5 \times 3$ or in general $ab = ba$.

2. The law of grouping, or Associative Law:

$$2 \times 3 \times 5 = 6 \times 5 = 2 \times 15 = 30.$$

or in general $abc = (ab)c = a(bc)$.

3. The law of distributing, or the Distributive Law:

$$2(3+7) = 2 \times 3 + 2 \times 7 = 6 + 14 = 20,$$

or in general $$a(b+c) = ab + ac.$$

52. We have also the law of signs for multiplying position and negative numbers.

Letting + stand for a positive number and — stand for a negative number, this Law of Signs may be stated

$$+ \cdot + = +$$
$$- \cdot - = +$$
$$+ \cdot - = -$$

or stated in words:

The product of two numbers of like sign is a positive number; the product of two numbers of unlike sign is a negative number.

53. If we multiply a^3 by a^5, we have

$$a\,a\,a \times a\,a\,a\,a\,a =$$
$$a\,a\,a\,a\,a\,a\,a\,a = a^8$$

that is $$a^2 \times a^5 = a^{3+5}$$

or in general, $$a^n\,a^m = a^{n+m}$$

This is called the Law of Exponents.

54. To multiply monominals together

$$2\,a^2 b \times 7\,a^3 b\,c = 2 \times 7\,a^{2+3} b^{1+1} c$$
$$= 14\,a^5 b^2 c$$

That is, multiply the numerical co-efficients and affix to each letter an exponent which is the sum of the exponents of that letter in the monominals to be multiplied.

Find the following products:

1. $7a^2b^3c \times 5a^7c$.
2. $3x^3y \times 9x^5y^2$.
3. $21a^2b^3 \times -35a^3b^2$.
4. $3t^2 \,.\, 4t^3 \,.\, 5t^3s \,.\, 2ts^3$.

5. $3n^2 \,.\, 2p^3 \,.\, 4n^3 \,.\, n \,.\, p.$

6. $-7a^2b \times .\, 6ab^3x^4 \,.\, -b^3x^3.$

7. $a^2 \,.\, \pi a^3 \,.\, \pi a \,.\, 2ab.$

55. To multiply polynominals.

$$\begin{aligned}(a+b)\ (c+d) &= (a+b)c + (a+b)d.\\ &= ac + bc + ad + bd.\end{aligned}$$

which may be stated:—**The product of two polynomials is the sum of the products of each term of the one with each term of the other.**

1. $x\,(x-1).$

2. $-x\,(x-2).$

3. $(n+2)\ (n-5).$

4. $(3n+1)\ (2n-3).$

$$(x^2+3x-5)\ (2x-3)$$

It is most convenient to arrange the work as in arithmetic. To test the accuracy of your work substitute in multiplicand, multiplier and product some convenient mumerical value.

$$\begin{array}{rrrrl} x^2 & +3x & -5 & & \qquad\text{Put } x=3 \quad 13\\ & 2x & -3 & & \qquad\qquad\qquad\quad\ \ 3\\ \hline 2x^3 & +6x^2 & -10x & & \\ & -3x^2 & -\ \ 9x & +15 & \\ \hline 2x^3 & +3x^2 & -19x & +15 & \qquad\qquad\qquad\quad 39 \end{array}$$

5. $(2t^2-3t+5)\ (2t-3)$

6. $(y^3+2y^2+y-1)\ (y^2-y+1)$

7. $(n^3+3n^2-2n+1)\ (n^2-2n+1)$

8. $(x^2-2xy+y^2)\ (x^2+2xy+y^2)$

9. $(4n-1/2)\ (5n+3/4)$

10. $(3x^4+6x^3y-9xy^3+12y^4)\ (2x^2-3xy+y^2)$

11. $(1+2n+3n^2+4n^3+5n^4)\ (2-3n+4n^2-5n^3)$

12. $(3x-2x^3+2x^4-7x^2+5)\ (3+3x^2+2x)$

It is rather better to have the terms arranged in an orderly manner for the work is then much easier.

It is wonderful how much easier work is if we take care to have everything in regular order. As has been said before we generally arrange a polynominal in ascending or descending powers of some letter.

14. $(3t-2t^2+5)\ (2-3t)$

15. $(h^3-3h+2)\ (2h^2-3)$

When terms of a regular order seem to be absent, it is often well to leave room for them.

16. $(4r^2-3r-1)\ (2r^3-3r+2)$

17. $n\ (n+2)\ (n-3)$

18. $(p-1)\ (p+1)\ (p^2+1)\ (p^4-p^2+1)$

19. $(x-1)\ (x+2)\ (x-3)\ (x+4)$

20. $(a-b)\ (b-c)\ (c-a)$

21. $(a^2+b^2+c^2-bc-ca-ab)\ (a+b+c)$

22. $(b+c)\ (b-c)+(c+a)\ (c-a)+(a+b)\ (a-b)$

23. $(a^2+ab+b^2)\ (a^2-ab+b^2)$

4 INVOLUTION.

56. 1. Multiply $(a+b)\ (a+b)\ (a+b)$

2. Multiply $2a^3 . 2a^3 . 2a^3 . 2a^3 . 2a^3$

We have seen that when all the factors of a product are equal we call the product a power of one of the factors and use the expotential notation for expressing it.

Thus the last two examples would be written

$$(a+b)^3, \quad (2a^3)^5$$

It would be nice if there were a shorter way of getting the results. Very well. Algebra is a sort of Aladdin's lamp which we have to rub and we get what we want.

57. For instance take

$$(ab)^4 = ab.ab.ab.ab = a^4b^4$$

or in general, $(ab)^n = a^n b^n$

which we call the Distributive Law of Involution.

It may be stated in words: The power of a product is equal to the product of that same power of the factors.

3. $(ax)^5 = ?$ a^5x^5
4. $(tpc)^7 = ?$ $t^7p^7c^7$
5. $(2x)^4 = ?$ $16x^4$
6. $(3x)^5 = ?$ $-243x^5$
7. $(12ha)^2 = ?$ $144h^2a^2$

58. Again take $(a^3)^4 = a^3.a^3.a^3.a^3 = a^{12} = a^{3\times 4}$ which is the second law of Involution.

8. $(a^5)^2 =$ a^{10}
9. $(x^2)^3 = ?$ x^6
10. $(t^5)^3 = ?$ t^{15}
11. $(y^3)^7 = ?$ y^{21}
12. $(x)^5 = ?$ x^5
13. $(3^2)^3 = ?$ 729

Use both laws and find

14. $(3a^4)^2 = ?$
15. $(2x^3)^5 = ?$
16. $(2ax^3)^4 = ?$

State a rule for finding any power of a monominal.

59. It is also important to observe the law of signs. Show that the following statements are true:

Any power of a positive number is positive.

An even power of a negative number is positive.

An odd power of a negative number is negative.

60. Perform the operations indicated.

1. $(3x)^3$
2. $(2ab^3)^4$

3. $(-2t^2)^3 = -8t^6$

4. $(-5n^2)^5 = -3125n^{10}$

5. $(3hx^2y^3)^3 = 27h^3x^6y^9$

6. $(-7m^3n^2p^3)^3$ $149m^9n^6p^9$

7. $(a^4c^3)^7 \cdot a^{28}c^{21}$

8. $(2/3\,h^2)^3$

9. $(3x^2y^4x^6w^3)^4$

10. $(a^nb^p)^m$

11. $(3x^ay^b)^a$

12. $(-10n^2t^3)^3$

13. $(x^2y^3)^n$ $x^{2n}y^{3n}$

14. $(-.1)^4$, $(-.1)^5$, $(-.2)^3$

15. $(-1)^{2n}$, $(-1)^{2n+1}$

16. $(3^n)^2$ $9n^2$

17. Write an expression that shall always be an even power of 2.

18. Write an expresion that shall always be positive for any value of n.

61. So much for monomials. Now let us take up the other example that started us on this investigation: $(a+b)^2$

This is the power of a binomial.

By multiplying find $(x+y)^2$, $(x+y)^3$, $(x+y)^4$

Four points are to be noted in each of these expansions What do you notice in connection with the four points suggested below?

1. The number of the terms. 2. The co-efficients of the terms. 3 and 4. The exponents of the x's and the y's.

1. In each expansion the number of the terms is one more than the exponent of the power.

2. The numerical co-efficients are called the binominal co-efficients and each power has its own peculiar set.

The co-efficients for a square are 1, 2, 1.
The co-efficients for a cube are 1, 3, 3, 1,
The co-efficients for a fourth are 1, 4, 6, 4, 1.

3. The x's begin in the first term with an exponent equal to the exponent of the power and the exponents in each succeeding term decrease by unity.

4. The y's begin in the second term with the exponent 1 and increase in each term by unity.

Note the symmetry of the expansions. It is one of the beauties of algebra that so many expansions display such orderly arrangement and symmetry.

62. The facts noted above enable us to write out at once the powers of a binominal without going thru the long multiplications.

Expand

1. $(a+b)^2$
2. $(x+h)^2$
3. $(y+3)^2$
4. $(a+2t)^2$
7. $(a+b)^3$
5. $(2n+3h)^2$
6. $(n^2+4)^2$
8. $(x+h)^3$
9. $(x+3)^3$

10. $(a+2t)^3$
11. $(2n+3h)^3$
12. $(n^2+5)^3$
13. $(a+b)^4$
14. $(x+h)^4$
15. $(p+2)^4$
16. $(a+2t)^4$
17. $(2s+3)^4$
18. $(x+y)^5$

How will $(x-y)^3$ differ from $(x+y)^3$?

19. $(a-c)^2$
20. $(t-s)^2$
21. $(n-3)^2$
22. $(x^2-6)^2$
23. $(x-1)^3$
24. $(c-d)^3$

25. $(x-2)^3$
26. $(2x^2-3)^3$
27. $(x-2)^4$
28. $(3t-5)^4$
29. $(3x^2-2y^3)^4$
30. $2x^2-\frac{1}{2x^2})^4$

The power of any polynomial may be obtained in the same way if first it be expressed as a binomial.

Thus $(m+n+p)^2 = ((m+n)+p)^2$
$$= (m+n)^2 + 2(m+n)\,p + p^2$$
$$= m^2 + 2mn + n^2 + 2mp + np + p^2$$

Apply to

31. $(a+b-c)^2$
32. $(a-b-c)^2$
33. $(m+n+p)^3$
34. $(x-y-z)^3$
35. $(x^2+2x+1)^2$
36. $(2x^3+3x+2)^3$
37. Simplify $3(n-1)+3(n-1)^2+(n-1)^3$
38. Simplify $(x-1)^3+7(x-1)^2+11(x-1)-2$
39. $2(x+h)^2+3(x+h)-2-(2x-1)\;(x+2)$
40. Put n inside the parenthesis:
$$n(n-3)$$
41. Put a inside the parenthesis:
$$a^3(a-b)^2$$
42. Express with all numbers inside the parenthesis
$$4x^2(2x-3)^2$$

63. There are several very interesting things about the powers of a binominal. We have found the binominal co-efficients to be:

for a 1st power $a+b$ to be 1, 1

for a square $(a+b)^2$ to be 1, 2, 1

for a cube $(a+b)^3$ to be 1, 3, 3, 1

for a fourth $(a+b)^4$ to be 1, 4, 6, 4, 1

for a fifth $(a+b)$ to be 1, 5, 10, 10, 5, 1

About two hundred and fifty years ago Pascal, a French mathematician, noticed that if you add the co-efficients of any vo consecutive terms in the expansion of one power the sum will be the co-efficient of a corresponding term in the expansion of the next higher power.

$$1+3=4, \quad 3+3=6, \quad 6+4=10, \text{ etc.}$$

Thus by what is called Pascal's triangle one can find the co-efficients for any power from the co-efficients of the preceding power.

Continue the table for the co-efficients of powers 6, 7, 8, 9, 10.

But Sir Isaac Newton, an Englishman who lived about two hundred years ago, was not satisfied with this. He wanted to know how to write out the expansion of any power without having to write out the expansion of the preceding power. So he set to work and discovered what is called The Binomial Theorem, or Binomial Formula, which gives outright the expansion of any power. That important formula is

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$$
$$+\frac{n(n-1)\ (n-2)}{2.3}\ x^{n-3}y^3 + \text{, etc.}$$

which will bear a good deal of study on your part.

Can you write out $(x+y)^{12}$, $(x+y)^{16}$?

5. DIVISION.

64. Division is the inverse of multiplication.

In multiplication we have two numbers to find the product. $3 \times 7 = ?$

In division we have the product and one of the numbers to find the other: $3 \times ? = 21$.

It is usual to express division as follows:

$$21 \div 3 = ? \text{ or } \frac{21}{3} = ?$$

We may look at it in this way: when we multiply a number by n, say, we introduce the factor n; when we divide a number by n we discard the factor n.

1. Divide ab by a
2. Divide xy by y
3. Divide abc by ab
4. Divide abc by abc
5. Divide 12a by 3
6. Divide 17xy by x
7. Divide 36st by 12t
8. Divide a^3n by a
9. Divide r^6 by r^3
10. $n^{10} \div n^7$
11. $(ax + bx) \div x$
12. $(ab - ac + ad) \div a$
13. $(2x^3 - 4x^2 + 6x) \div 2x$
14. $(x^2 - 1) \div (x + 1)$
15. $(x^2 + 2xy + y^2) / (x + y)$
16. $(x^2 - 3x + 2) / (x - 2)$

65. In the division $n^{10} \div n^7 = n^3$ we have discarded n seven times. The number of n's left may be found by subtracting the exponent of the divisor from the exponent of the dividend.

$$n^{10} \div n^7 = n^{10-7}$$

In general $$a^m \div a^n = a^{m-n}$$

We can be certain of the truth of the general law if we make use of the fact that division is the inverse of multiplication. The divisor times the quotient must equal the dividend: Apply to

1. $-8x^2y^3 / 2x$

State the law of signs for division.

2. $25a^7b^3c^3 / 5a^5b^2c$
3. $39m^5n^{10} / -3mn^3$
4. $a^6b^6c^7 / -a^5b^2c^3$
5. $(32k^2n^3 - 16k^2n^5) / 8kn^2$
6. $(36x^3y^4 - 42x^5y^6) / -6x^2y$
7. $(10x^2y - 15x^3y^3 + 5x^3y) / -5x^2y$

State a rule for dividing an expression by a monomial.

In dividing $10x^2y$ by $-5x^2y$ by the exponential method we get $-2x^0y^0$. What does x^0 mean? Going back to the method of division by discarding factors

$$x^3 \div x^3 = 1$$

The result of the expotential method must be the same.

$$x^2 \div x^2 = x^{2-2} = x^0$$

therefore $$x^0 = 1.$$

The zero power of any number is always unity. This may seem a little curious to you. Look at it in this way,

$$a^3 = 1 . a . a . a$$
$$a^2 = 1 . a . a$$
$$a^1 = 1 . a$$
$$a^0 = 1$$

8. Find he value of a^0, 7^0, 3^0, 1^0.

9. $(.16a^3 + .06a^2 + .2a) / .2a$

10. $(27x^4 - 9x^3 - 12x^2) / 3x^2$

66. Thus far we have been considering division by a monomial. Let us now consider division by a polynomial.

We proceed as in long division in arithmetic.

Divide 8673 by 21, If $10 = t$, we may write 8673 as $8t^3 + 6t^2 + 7t + 3$, and the division may be written:

$$\begin{array}{rl}
413 & \\
8673 & (\underline{\ 21\ } \\
84 & \\
\hline
27 & \\
21 & \\
\hline
63 & \\
63 &
\end{array}
\qquad\qquad
\begin{array}{rl}
4t^2 + t + 3 & \\
8t^3 + 6t^2 + 7t + 3 & (\underline{\ 2t + 1\ } \\
\underline{8t^3 + 4t^2} \qquad\qquad & \\
2t^2 + 7t \qquad & \\
\underline{2t^2 + t} \qquad & \\
6t + 3 & \\
\underline{6t + 3} &
\end{array}$$

Apply to the following

1. Divide $x^3 + 9x^2 + 26x + 24$ by $x + 2$

2. Divide $6m^3 - m^2 - 14m + 3$ by $3m^2 + 4m - 1$

3. $(n^4 + n^3 + 7n^2 - 6n + 8) / (n^2 + 2n + 8)$

State a rule for dividing by a polynomial.

4. $(4 + 12a + 13a^2 + 6a^3 + a^4) / (2 + 3a + a^2)$

5. $(x^4 - x^3y + 2x^2y^2 - xy^3 + y^4) / (x^2 - xy + y^2)$

6. $(4n^2 - n^3 + 4n + 2n^4 - 3) / (3 + n^2 - n)$

Care should be taken that the expressions are arranged in either ascending or descending powers of a letter before the divsion is performed. Why?

It is better to keep like terms under each other. To gain this it is often well to leave a vacant space.

7. $(5k^3 - 3 - 5k + 6k^5 - 3k^4) / (3k^2 - 1 - 2k)$

8. $(6h^5 - 9h^4 + 10h^3 - 18h^2 - 5) / (3h^2 + 5)$

9. $(a^4 - 1) / (a + 1)$

10. $(x^4 + x^2y^2 + y^4) / (y^2 - xy + x^2)$

11. $(n^6 - p^6) / (n^2 + np + p^2)$

12. $(\frac{1}{8}a^3 - \frac{9}{4}a^2x + \frac{27}{2}ax^2 - 27x^3) / (\frac{1}{2}a - 3x)$

Divide $x^2 - 3x + 2$ by $x + 2$

In this case there is a remainder 12 which is not exactly divisible by the divisor x. The division is inexact. In expressing the result we follow the same plan as in arithmetic and place this remainder over the divisor (that is, we merely indicate the division) and add this to the quotient:

$$(x^2 - 3x + 2) \div (x + 2) = x - 5 + \frac{12}{x + 2}$$

The indicated division $12/(x + 2)$ is called a fraction. In truth any indicated division is called a fraction, the dividend being called the numerator and the divisor the denominator. In fact this is the most common and the most convenient way of expressing or indicating a division. Remember that whenever a fraction comes up, and they are always coming up in our mathematical work,—remember that they are but indicated divisions. We will take up the special study of fractions later.

In cases where the division is inexact the division should be carried on until the highest exponent of the remainder is less than the highest exponent of the divisor.

13. Divide $x^2 + 1$ by $x + 1$

14. $(x^3 + x - 25) / (x - 3)$

15. $(h^6 - 11h^3 + 21) / (h + 5)$

16. $(x + 1) / (x - 1)$

17. $(1 + x) / (1 - x)$. How long can this division be carried on?

18. $(1 + 3x^3) / (1 - 2x)$ Stop after four terms have been obtained.

67. In all the examples of division we have considered except the last few the division has been exact. It will be more convenient to consider cases of inexact division later in connection with fractions. We will postpone such consideration until Chapter 7. In the meantime we will take up some other matters which will be of great aid to us when we do undertake the study of fractions.

68. The sixth important operation used in algebra is the extraction of roots of numbers. But we will postpone this subject until a little later. What knowledge you have of it will be sufficient for your present needs.

CHAPTER 6.

FACTORS AND MULTIPLES.

70. We have found that a knowledge of factoring enabled us to simplify certain computations and to solve certain kinds of equations. It will also be of service to us in working with fractions, a pleasure which you are promised in the next chapter. In view of the usefulness of factoring in dealing with fractions it is well for us first to become better acquainted with the possibilities of factoring.

In Chapter 2 we considered four types of expressions that can be factored. In this chapter we will add to these a few more types tho by no means will we exhaust the subject. A mastery of the types we are to consider will often serve you in good part in simplifying many a complicated expression and in leading you over difficulties that otherwise would be unsurmountable.

To factor an expression is to express it as a product. Thus:

$$a^2-9=(a+3)\ (a-3)$$

Not all expressions can be so expressed. It is very important that one should be able to determine whether a given expression is factorable or not. One must learn to detect, at a glance, the ear-marks of at least the simpler types.

It will be well, now, for you to review the four types given in Chapter 2, and apply to the examples given below.

The four types considered were:

1. A common factor.
2. The difference of two squares.
3. A trinomial square.
4. A quadratic form.

1. $a^2 - 121$
2. $x^2 - b^2c^2$
3. $x^2y + xy^2 + xy$
4. $x^2 - 5x + 6$
5. $y^2 - 3y - 88$
6. $3x^2 - 12x + 12$
7. $n^3 + 3n^2 + 2n$
8. $t^2 - 10t + 25$
9. $\frac{1}{8}\pi a^2 + \frac{1}{8}\pi b^2$
10. $a^3 + a - 2a^2$
11. $7nmp^2 - 7nm^3$
12. $R^2 - 10R + 21$
13. $R^2t - t^3$
14. $pv^2 - pv^3$
15. $h^2 + k^2 - 2hk$
16. $x^6 + 8x^4 + 16x^2$
17. $16.1\,(9.5^2 - 8.5^2)$
18. $\pi\,3.7^2 - \pi\,2.9^2$

71. Factoring is often assisted by grouping terms.

A special case of Type 1 often appears in which there exists a common binomial factor which may be revealed by grouping certain terms together..

A careful study of the form

$$ax + ay + bx + by$$

will show that it can be reduced to the form

$$a(x + y) + b(x + y)$$

where $x + y$ is a common binominal factor, and factoring we have $(x + y)\ (a + b)$.

So also $t^3 - 2t^2 - 3t + 6 = t^2(t - 2) - 3(t - 2)$

$$= (t - 2)\ (t^2 - 3)$$

Factor.

1. $a^2 + ab + ac + bc$
2. $3x^3 + 2x^2 + 3x + 2$
3. $ax^2 - bx^2 + ay^2 - by^2$
4. $x^2 + 2ax + 3bx + 6ab$
5. $h^3 + h^2k + hk^2 + k^3$
6. $acn^2 - bcn + adn - bd$
7. $5x^3 - x^2 - 5x + 1$
8. $6t^2 + 3ts - 2at - as$

72. Another important case of grouping reveals the difference of two squares.

$$\begin{aligned} a^2+2ab+b^2-c^2 &= (a^2+2ab+b^2)-c^2 \\ &= (a+b)^2-c^2 \\ &= (a+b+c)\ (a+b-c) \end{aligned}$$

Factor.

1. $x^2-2x+1-y^2$
2. $x^2-a^2+2ab-b^2$
3. $1+2ab-a^2-b^2$
4. $x^2+y^2-z^2-2xy$
5. $4-n^2-p^2-2pn$
6. $a^2+b^2-c^2-d^2+2ab+2cd$
7. $4-4n+n^2-n^4$
8. $h^4+4h^3+4h^2-9$

73. Types of factorable expressions are often revealed by special forms of multiplication.

Multiply a^2+ab+b^2 by $a-b$ and get a^3-b^3. This furnishes another easily distinguished type for factoring.

Type 5. a^3-b^3. The difference to two cubes.

The type formula is

$$a^3-b^3=(a-b)\ (a^2+ab+b^2)$$

Stated in words this is: The difference of the cubes of two numbers equals the product of the first minus the second, and the square of the first, plus the product of the two, plus the square of the second.

Expressions of the form a^3-b^3 can now be easily factored.

Factor.

1. a^3-b^3
2. c^3-8
3. x^3-125
4. n^3-64
5. r^3-9
6. h^3-k^3
7. $x^3y^3-a^3b^3$
8. $27n^3-8p^3$
9. $1-64t^3$

10. $125 - 27H^3$	13. $x^4y - xy^4$
11. $n^4t - nt^4$	14. $4c^5 - 32b^3c^2$
12. $\pi R^3 - \pi r^3$	15. $a^3 + b^6$

74. Again by multiplication we have

$$(a^2 - ab + b^2)\ (a + b) = ?$$

This furnishes another type

Type 6. $a^3 + b^3$. The sum of two cubes.

And the formula is

$$a^3 + b^3 = (a + b)\ (a^2 - ab + b^2)$$

In what respects does this type differ from the preceding type?

Factor.	9. $64a^3 + 729$
1. $x^3 + y^3$	10. $25x^3 + 200y^3$
2. $c^3 + 8$	11. $n^4t + nt^4$
3. $x^3 + 125$	12. $1 + 16s^3$
4. $n^3 + 27p^3$	13. $16 + 54n^3$
5. $x^3y^3 + t^3$	14. $1728 + 1331x^3$
6. $1 + 125h^3$	15. $n^5t^2 + n^2t^5$
7. $1 + s^3$	16. $a^6 + b^6$
8. $t^3 + 64$	17. $a^9 + b^9$

75. We have seen that the quadratic form $x^2 + bx + c$ can be factored when b is the sum of two numbers whose product is c.

We might ask the question, when, if ever, is it possible to factor the more general quadratic form

$$ax^2 + bx + c$$

In the multiplication

$$\begin{array}{rrr}
2x+3 & & \\
5x+7 & & \\
\hline
10x^2+15x & & \\
+14x & +21 & \\
\hline
10x^2+29x & +21 &
\end{array}$$

that 10, the co-efficient of x^2 is 2×5
that 21, the last term is 3x7
that 29, the co-efficient of the x is $3 \times 5 + 2 \times 7$
or in general

$$(mx+n)\ (px+q) = mpx^2 + (np+qm)x + nq.$$

consequently if $ax^2 + bx + c$ is factorable, the factors must be of the form $(mx+n)\ (px+q)$. Where $a = mp$, $c = nq$, and $b = np + qm$.

If then ac can be separated into two factors whose sum is b, the expression can be factored. This is then the condition upon which an expression of the form given above can be factored.

An illustration will make this a little clearer.

Factor $10x^2 + 19y + 6$.

Find two numbers whose product is 6×10, (60) and whose sum is 19. These are 15 and 4.

Separate the middle term into terms with these numbers as coefficients:

$$\begin{aligned}
10x^2 + 19x + 6 &= 10x^2 + 15x + 4x + 6 \\
&= 5x(2x+3) + 2(2x+3) \\
&= (5x+2)\ (2x+3)
\end{aligned}$$

so also

$$\begin{aligned}
10n^2 - n^2 - 21 &= 10n^2 - 15n + 14n - 21 \\
&= 5n\ (2n-3) + 7(2n-3) \\
&= (2n-3)\ (5n+7)
\end{aligned}$$

Thus we have another type.

Type 7. $ax^2 + bx + c$. A general quadratic form.

This is a more general case including Type 4 as a special case. The test of factorability is, b must be the sum of two numbers whose product is ac.

Just as in Type 4, this is not always possible. At least it appears so from our present knowledge. But we must learn to be modest, for we have not yet reached the limit of algebraic knowledge. When your knowledge has been extended a little more you will find that all such expressions can be factored. That is something for you to look forward to.

Be very sure, however, before you give up factoring an expression of this kind that you have exhausted all the tests at hand.

Factor

1. $2x^2 + 11x + 5$.
2. $2x^2 + 11x - 8$.
3. $4c^2 + 23c + 15$.
4. $6 + 5a - 6a^2$.
5. $3x^2 + 7x - 6$.
6. $4 - 5x - 6x^2$.
7. $2h^2 + h - 28$.
8. $10 - 5t - 15t^2$.
9. $2f^2 + 3f - 2$.
10. $2h^2 - 5hk - 3k^2$.
11. $24x^2 - 29xy - 4y^4$.
12. $6y^2 + 7y - 3$.
13. $3a^2 + 19ab - 14b^2$.
14. $12x^2 - 23xy + 10y^2$.
15. $7 + 10n + 3n^2$.
16. $x^2 + 43xy + 390y^2$.
17. $3x^2 + 10x + 3$.
18. $9a^2 + 16b^2 - 24ab$.
19. $4t^2 + t - 14$.
20. $7x^4 - 15x^2y^2 - 18y^4$.

76. We might ask the question: Is is possible to factor a more general cubic form such as

$$x^3 - 6x^2 + 11x - 6.$$

or an expression of still higher degree,

$$5x^4 + 3x^3 - 4x + 5.$$

Such polynomials can often be easily factored and we will now begin an investigation which will eventually show us how this can be done.

77. Consider the algebraic expression

$$x^2+2x-3.$$

If $x=0$, it becomes $=-3$
$x=1$, it becomes $=0$
$x=2$, it becomes $=5$
$x=3$, it becomes $=12$
$x=4$, it becomes $=21$ and so on.

That is to say, the value of x^2+2x-3 depends upon the value given to x.

We say that x^2+2x-3 is a function of x.

We say that any algebraic expression is a function of a number if its value changes as the value of the number changes.

We often use the notation $f(x)$ to denote that an expression is a function of x.

If $f(x)=x^2-2x+3$
then $f(y)=y^2-2y+3$
and $f(a)=a^2-2a+3$
and $f(4)=4^2-2.4+3$

We read $f(x)$, the f - function of x

$f(a)$, the f - function of a, or more simply the function of x, or the f of x.

Note the functions given above are all of the same form.

1. If $f(x)=x^3-x+2$, write out $f(n)$, $f(h)$, $f(3)$, $f(2)$, $f(0)$, $f(5)$.

2. If $f(x)=2x^2-7$, find $f(k)$, $f(7)$, $f(-1)$.

3. If $f(y)=(y-2)^2+3y$, find $f(x)$, $f(t)$, $f(3n)$, $f(3)$, $f(-2)$.

Now consider $f(x)=x^3-2x^2+4x-5$.

Divide x^3-2x^2+4x-5 by $x-2$: the quotient is x^2+4 and the remainder is 3.

Now we know that

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\frac{D}{d} = Q + \frac{R}{d} \text{ that is } D = Qd + R$$

whence we have for the special case above.

$$f(x) = x^3 - 2x^2 + 4x - 5 = (x^2 + 4)(x - 2) + 3$$

Now let us put $x = 2$ in the expressions above and get

$$f(2) = \overset{8}{2^3} - \overset{8}{2.2^2} + \overset{8}{4.2} - \overset{5}{5} = (2^2 + 4)(\cancel{2} - \cancel{2}) + 3$$

$$f(2) = 3$$

$$\text{that is} \quad f(2) = R$$

This gives a short way of finding the remainder when the divisor is a binominal of the form $x - a$.

Find the remainder of $(x^2 - 2x + 1) / (x - 2)$,

$$f(x) = x^2 - 2x + 1$$

$$R = f(2) = 2^2 - 2.2 + 1 = 1.$$

Find the remainder when $x^3 - 2x^2 + 3x - 2$ is divided by $x - 3$.

$$f(x) = x^3 - 2x^2 + 3x - 2.$$

$$R = f(3) = 3^3 - 2.3^2 + 3.3 - 2 = 16.$$

Verify these results by actual division.

The interesting fact just noticed and used in this short method of finding the remainder may be stated in general terms as follows:—

If $f(x)$ be divided by $x - a$ the remainder will be $f(a)$.

This is known as the *Remainder Theorem*.

Find the remainders in the following

1. $(x^4 - x^3 + x^2 - x + 1) \; / \; (x - 1)$.
2. $x^4 - x^3 + x^2 - x + 1) \; / \; (x - 2)$.

3. $(x^3 - 3x^2 + 2x - 3) \ / \ (x - 3)$.

4. $(x^3 - 3x^2 + 2x - 3) \ / \ (x - 5)$.

$x + a$ may be written $x - (-a)$.

5. $(x^3 - 3x + 2) \ / \ (x + 2)$.

6. $(x^5 - x^3 - 5) \ / \ (x + 3)$.

78. If the remainder obtained in any division is zero then the divisor is a factor of the dividend. For example in the division

$$x^3 - x^2 - x - 2) \ / \ (x - 2)$$

$$R = f(2) = 0.$$

It follows that $x - 2$ is a factor, the other factor can be obtained by divisions.

This can be stated in general as:—

If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

This is known as the ***Factor Theorem.***

79. This important theorem may now be used in determining whether a given function of the form noted in Art. 76 has a binomial factor of the form $x - a$. In order that $x - a$ may be a factor the a must be a divisor of the last term of the function to be factored. Why?

Let us illustrate with an example.

$$f(x) = x^3 - 4x^2 + x + 6.$$

The possible binomial factors are:

$x - 1$, $x - 2$, $x - 3$, $x - 6$, $x + 1$, $x + 2$, $x + 3$, $x + 6$.

Try these in turn.

Try $x - 1$, $f(1) = 4$, $x - 1$ is not a factor.

Try $x - 2$, $f(2) = 0$, $x - 2$ is a factor.

The other is found by division.

$$x^3 - 4x^2 + x + 6 = (x - 2)\ (x^2 - 2x - 3),$$
$$= (x - 2)\ (x - 3)\ (x + 1).$$

Factor

1. $x^3 + 6x^2 + 13x + 10.$
2. $n^3 + 5n^2 + 3n - 9.$
3. $x^3 - x^2 - 2x + 2.$
4. $x^4 - 4x^3 + 8x^2 - 8x - 21$
5. $x^3 + 3x^2 + 4x + 4.$
6. $x^5 - 3x^3 + 8.$
7. $x^3 - 7x + 6.$
8. $t^3 - 4t + 3.$
9. $x^3 - 3x - 1.$

SUMMARY OF TYPES.

80. 1. A common factor

$$ab + ac = a(b + c)$$

2. The difference of two squares

$$a^2 - b^2 = (a + b)\ (a - b).$$

3. A trinomial square

$$a^2 + 2ab + b^2 = (a + b)^2.$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

4. A quadratic form

$$x^2 + bx + c \quad \text{when} \quad c = np,\ b = n + p$$

5. The difference of two cubes

$$a^3 - b^3 = (a - b)\ (a^2 + ab + b^2).$$

6. The sum of two cubes

$$a^3 + b^3 = (a + b)\ (a^2 - ab + b^2).$$

7. The general quadratic form

$$ax^2 + bx + c \quad \text{when} \quad ac = np,\ b = n + p.$$

8. The general polynomial in powers of one letter.

$$x^n + px^{n-1} + qx^{n-2} + \ldots. + sx + t.$$

81. Sometimes expanding an expression which does not seem to be factorable will reduce it to the form of one of the types.

$$(a + b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab.$$
$$= a^2 - 2ab + b^2.$$
$$= (a - b)^2.$$

The example just given also illustrates the fact that it is not always possible to get the original expression by multiplying together its factors, altho the product so obtained is always equal to the original expression.

Factor

1. $4ad + (a - d)^2$.
2. $(7.2 + 3.4)^2 - 4 \times 7.2 \times 3.4$.
3. $(x + 1)^2 - 5x - 29$.
4. $a^2 (a^2 - 24) + 63$.
5. $x^2 (x - 2y) - y^2 (y - 2x)$.
6. $k (k - 1) - c (c + 1)$.
7. $m (m^2 - 1) - n (n^2 - 1)$.
8. $n^2 - (n - a)^2$.

82. Expressions like

$$2ab - a^2 - b^2.$$
$$2 - x - x^2.$$
$$2 - 3x - 2x^2.$$

may give a little trouble in factoring. The matter is really very simple if one looks at it in the right way.

$$\begin{aligned} 2ab - a^2 - b^2 &= -(-2ab + a^2 + b^2). \\ &= -(a^2 - 2ab + b^2). \\ &= -(a - b)^2 = - (a - b)\ (a - b). \\ &= (a - b)\ (b - a) \end{aligned}$$

Note that the outlying minus sign may be introduced into one of the factors.

$$\begin{aligned} 2 - x - x^2 = - (x^2 + x - 2) &= -(x + 2)\ (x - 1). \\ &= (x + 2)\ (1 - x). \end{aligned}$$

So also

$$\begin{aligned} 2 - 3x - 2x^2 &= - (2x^2 + 3x - 2). \\ &= ? \end{aligned}$$

83. GENERAL EXERCISES IN FACTORING.

First determine to what type the expression belongs. You should then have little difficulty in factoring it. Before you say that any of the expressions cannot be factored be very sure that you have tried all the tests at your command. Even then it is not always safe to say that it cannot be factored. For there are many types of factorable expressions that we have not considered in this chapter. We have taken up the more common and simpler ones. You will meet with others in your study of mathematics.

Factor the following.

1. $2\pi R - 2\pi r$.
2. $\pi R^2 - \pi r^2$.
3. $x^2 - x - 20$.
4. $a^3 - 8$.
5. $R^2 - 5R + 6$.
6. $497^2 - 395^2$.
7. $125 + 64x^3$.
8. $5x^3 - 10x^2 + 5x$.
9. $a^2 + ax + x + a$.
10. $n^3 - t^3 - n + t$.
11. $4h^2 + 8h + 3$.
12. $x^4 + x^2 + x^3 + 1$.
13. $4an^2 + 12an^3$.
14. $2y^2 - 11y + 15$.
15. $25 - 49h^2$.
16. $64t^3 + 27s^3$.
17. $n^2 - (n - a)^2$.
18. $\frac{wt}{2}x - wx\frac{x}{2}$
19. $4ad + (a - d)^2$.
20. $(a + d)^2 - 4ad$.
21. $6x^2 - 17x + 5$.
22. $6x^2 + 13x - 5$.
23. $6x^2 + 17x + 5$.
24. $a^2 - ac + bc - b^2$.
25. $81 + 18b + b^2$.
26. $m^2 - 9mn + 20n^2$.
27. $x^{2n} - y^{2n}$.
28. $x^{3n} - y^{3n}$.
29. $9x^n - y^{n+2}$.
30. $a^2 - b^2 - (a - b)^2$.
31. $n^2 + 14n + 33$.
32. $H^2 + 3H + 2$.
33. $T^2 - 2aT$.
34. $2\pi rh + 2\pi r^2$.
35. $a^2 + 2ab + b^2 - 1$.

36. $a^2 - b^2 + 2bc - c^2$.

37. $1 - n^2 - t^2 - 2nt$.

38. $x^4 - 2x^3 + 2x - 1$.

39. $x^3 - 3x^2 + 4$.

40. $a^3b - ab^3 + a^2b + ab^2$.

41. $4n^2 + 23n + 15$.

42. $65 + 8xy - x^2y^2$.

43. $4x^3 - 6x^2 + 2$.

44. $\pi r^2.2r - \pi r^2.\frac{2r}{3}$

45. $t^4 + s^4$.

46. $t^4 - s^4$.

47. $12x^2 - 30x + 12$.

48. $3(n-1) + 3(n-1)^2 + (n-1)^3$.

49. $9t^2 - 12ts + 16s^2$.

50. $9t^2 - 24ts + 16s^2$.

51. $3(x-4)^2(x+2)^2 + 2(x+2)(x-4)^3$.

52. $\frac{1}{2}wt(\frac{1}{2}t - x) - \frac{1}{2}w(\frac{1}{2}t - x)^2$.

53. $(x-5)^2 - 8(x-5) - 20$.

54. $x^4 + x^3y + xz^3 + yz^3$.

55. $ab - a + b - 1$.

56. Find value when $a = 1$. $b = 3$. $c = 5$. $r = 7$.
$2\pi ar + 2\pi br + 2\pi cr + 2\pi r^2$.

57. Find value
$2\pi R^2 - 2\pi r^2 + 2\pi Ra + 2\pi ra$.
when $R = 3.1$ $r = 2.1$ $a = 3$.

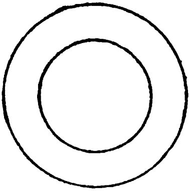


Fig 5

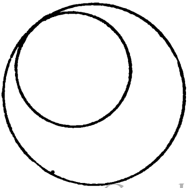


Fig 6

58. Find the area of the ring furnished you.

59. Find the area of the crescent furnished you.

60. A flag-staff, AB, 50 ft. high was broken off at a point C. the broken part resting on C reached the ground at D, 30 feet from the base of the staff. Find the length of the staff still standing.

Solve the equations (by factoring).

61. $x^2 - 7x = 10.$

62. $y^2 - 16 = 0.$

63. $6x^2 - 7x = 3.$

64. $8p^2 + p = 30.$

65. $3h^2 = 5h.$

66. $x^3 - 4x = 4 - x^2.$

67. $h^3 - 3h^2 = 4h - 12.$

68. $2n^3 + 5n = 7n^2.$

69. $4t^2 - 12t = -9.$

70. $x^3 + bx^2 = a^2x + a^2b.$ Solve for x.

71. $x^4 - 15x^2 + 10x = -24.$

72. $x^2 - 2ax = b^2 - a^2.$

HIGHEST COMMON FACTOR.

84. Any factor that appears in each of several expressions is called a common factor of those expressions.

The product of all the factors common to two or more expressions is called the Highest Common Factor of those expressions.

The H. C. F. of ax^3y^4, bx^2y. $a^2x^5y^3$, is x^2y.

The H. C. F. of $a\,(x + 6^2)$, $b(x^2 - 36)$. is $(x + 6.)^2$

It is to be noticed that since the H. C. F. contains only the common factors it is therefore a divisor of all the expressions. It is then the factor of the highest degree common to the expressions. This is the most important property of the H. C. F. and it is to this property that it owes its name.

The H. C. F. can be found by an inspection of the factored forms of the expressions.

Find the H. C. F.

1. 24, 36. 42, 84.
2. $3x^2y^2$, $12xy^2$, $24x^3y$.
3. $27a^3b^7c^6$, $9a^8b^5c^7$, $18a^7b^8c$.
4. $(a-x)c$, a^2-x^2, a^3-x^3.
5. x^2-16, x^2-5x+4.
6. n^4-1., n^3-1, n^2-1.
7. x^2-3x+2, x^2+2x-3.
8. $x^2-9x+14$, $2x^3-5x^2-441$.
9. k^3-n^3, k^4-n^4.
10. $(a^2+y^2)^2$, a^4-y^4.
11. i^5-ik^4, $i^3+ik+i^2k+k^2$, i^6-k^6.
12. $x^3+3x^2y+2xy^2$, $x^4+6x^3y+8x^2y^2$.
13. f^3+f^2+f-3, f^3+3f^2+5f+3.

LOWEST COMMON MULTIPLE.

85. If a number N be multiplied by a number M, the product NM is called a multiple of N.

A common multiple of two or more numbers is a number which is a multiple of each.

It follows directly from the definition that a common multiple is divisible by each of the numbers of which it is a common multiple. And it is to this property of a common multiple that it owes its importance in algebra, especially in the addition of fractions.

The product of the different prime factors of two or more expressions, each factor being taken as many times as it appears in any one of them, is called the Lowest Common Multiple of those expressions

It is the expression of lowest degree that is exactly divisible by each of the expressions.

The L. C. M. can be found by inspection of the factored forms of the expressions.

Find the L. C. M.

1. 36, 42, 12, 27.
2. $5a^2b^4c^3$, $4abc^5$.
3. a^2c, bc^2, cb^2.
4. $7a^2$, $2ab$, $3b^2$.
5. $2xy$. $3x^2y$, $4xy^2$.
6. $24a^4b^2c^3$, $42a^6b^3c^2$.
7. $(x-y)^2$, x^2-y^2, x^2.
8. $x-1$, x^2+x+1, x^3-1.
9. $x-3$, $x-4$, $x^2-7x+12$.
10. $1-2a$, $1+2a$, $4a^2-1$.
11. a^3-b^3, a^3+b^3. a^2+ab+b^2.
12. x^2+5x+6, x^2+3x+2, x^2+4x+8.
13. n^2-b^2, $n+b$, $(n^2-b^2)^2$.
14. $6h^2+54$, $3h-9$ $3h^2-27$.
15. $12x^2+3x-42$, $12x^2+30x+12$, $32x^2-40x-28$.

CHAPTER 7.

FRACTIONS.

86. Let us return now to the broken thread of our discussion of inexact division which we left at the close of Chapter 5. We had found that in some cases of division there was a remainder that could not be divided exactly by the divisor. The division could therefore only be expressed. This expressed division was called a fraction.

So in the case $3x^2y \div xy^2$, or to use the better notation

$$\frac{3x^2y}{xy^2}$$

In dividing by a product of several factors the best way is always to divide by the factors one at a time, in other words we cast them out one at a time. In the case above we cast out an x and a y. There is still a y to be cast out, but as there is none in the dividend this cannot be done and the division can only be expressed. We have then $\frac{3x}{y}$ which we call a fraction. The form $\frac{3x^2y}{xy^2}$ is so compact and convenient that we give the same name to it. In fact we define a fraction as being an expressed division, the dividend being called the numerator and the divisor the denominator. We have $\frac{3x^2y}{xy^2} = \frac{3x}{y}$ where as much of the division as was possible has been performed. It will be noted that the terms of the second fraction contain no common factor, that is they are prime to each other. We may consider that the second fraction has been obtained from the first by casting out (dividing) an x and a y from both numerator and denominator. So that the value of the fraction has been unchanged by dividing both numerator and denominator by the same number. We have here stumbled upon one of the important laws of fractions which we will consider later.

87. What would be the effect upon a fraction if we were to multiply its numerator by any number?

Take the fraction $\frac{3}{5}$ and multiply its numerator by 2. What effect?

Is this true for every fraction? Let us see. Suppose we multiply the numerator of the fraction $\frac{a}{b}$ by c.

By definition $\frac{ac}{b} = ac \div b.$

$= a \times c \div b.$ Why?

$= a \div b \times c.$ Why?

$= \frac{a}{b} \times c.$ Why?

We have then the first law of fractions:

Law 1. If the numerator of a fraction be multiplied by any number the fraction is multiplied by that number.

88. What will be the effect if a factor is introduced into the denominator of a fraction?

Take the fraction $\frac{a}{b}$

$\frac{a}{bc} = a \div bc.$ Why?

$= a \div b \div c.$ Why?

$= \frac{a}{b} \div c.$ Why?

Which is the second law of fractions.

Law 2. If the denominator of a fraction be multiplied by any number the fraction is divided by that number.

89. Now what will be the effect if the same factor be introduced into both numerator and denominator?

$$\frac{ac}{bc}=\frac{ac}{b}\div c$$

$$=\frac{a}{b}\times c\div c$$

$$=\frac{a}{b}$$

and the third law is:

Law 3. If the numerator and denominator of a fraction be multiplied by the same number the value of the fraction is unchanged.

90. *Important:*—It is to be noted that the statement of the three laws given above are true also if the words, "multiplied" and "divided" are interchanged.

State the three laws with those words interchanged, and show that they are true.

91. From the results of the last few articles it is at once seen that there are two ways of multiplying a fraction, what are they?

There are also two ways of dividing a fraction, what are they?

There are also two ways in which both numerator and denominator may be altered without changing the value of the fractions, what are they?

92. Fractions will be considered under eight distinct topics:—

1. Reduction or transformation.
2. Addition and subtraction.
3. Multiplication.
4. Involution (finding a power).

5. Division.
6. Evolution (finding a root).
7. Complex fractions.
8. Ratio.

93. A fraction is said to be in its lowest terms if the numerator and denominator are prime to each other, that is, if they have no factor in common.

A fraction is reduced to its lowest terms exactly as in arithmetic, by dividing both terms by the same number, that is by casting out factors common to the numerator and denominator.

$$\frac{ab}{ac}=\frac{\cancel{a}b}{\cancel{a}c}=\frac{b}{c}\ ,\quad \frac{ab+ac}{ad}=\frac{\cancel{a}b+\cancel{a}c}{\cancel{a}d}=\frac{b+c}{d}$$

Reduce the lowest terms.

1. $\frac{3a^2c}{5ab}$
2. $\frac{6a^2b^3}{12a^3b^2c}$
3. $\frac{4x^5y^7z}{6x^5y^2z}$
4. $\frac{28a^9b^4c^5}{42a^6bc^6d^7}$
5. $\frac{7x^3y}{x^2-xy}$
6. $\frac{a^3-b^3}{a^2+b^2}$
7. $\frac{t^3+2t^2}{3t^5+6t^4}$
8. $\frac{a^2-b^2}{a^4-b^4}$
9. $\frac{r^2-2r+1}{r^2-1}$
10. $\frac{3a+3b}{m(a+b)}$
11. $\frac{Rr^2-R^3}{\pi R^2-\pi rR}$
12. $\frac{3h^2k-3k^3}{2h^3k-2k^4}$
13. $\frac{nx^3y-nxy^3}{nx^4y-nx^2y^3}$
14. $\frac{18a^4b^2-27a^6}{10b^2-15a^2}$
15. $\frac{n^3-p^3}{n^2+np+p^2}$
16. $\frac{a^2-b^2}{b-a}$

17. $\dfrac{am - an - m + n}{am - an + m - n}$

18. $\dfrac{x^2 - 4x + 4}{x^2 - 5x + 6}$

19. $\dfrac{x^2 + y^2 - z^2 + 2xy}{x^2 - y^2 - z^2 + 2yz}$

20. $\dfrac{p^2 + 2pq + q^2 - r^2}{p + q + r}$

21. $\dfrac{s^2t + st^2 + t^3}{s^3 - t^3}$

22. $\dfrac{12y^3 - 7y^2 - 12y}{9y^3 + 6y^2 - 24y}$

23. $\dfrac{x^{n+2} - x^n}{x^{n+1} - x^n}$

24. $\dfrac{x^3 - 5x^2 + 7x - 3}{2x^3 - 5x^2 - 4x + 3}$

94. A fraction whose numerator is of the same degree as its denominator or of a higher degree, many be reduced to an integral expression composed of an integral expression and a fraction. Such a fraction is called an improper fraction. Since the fraction is an indicated division, this reduction is made by simply performing the division.

Reduce to mixed numbers.

1. $\dfrac{3x^2 + 2x + 1}{x + 4}$

2. $\dfrac{n^3 - 3n^2 + 1}{n^2 - n}$

3. $\dfrac{4a^2 + 6ab + 9b^2}{2a + 3b}$

4. $\dfrac{x - 1}{x + 1}$

5. $\dfrac{1 - x}{1 + x}$

95. The reverse operation of reducing a mixed number to the form of a fraction is of course done by multiplication.

1. $1 + x + \dfrac{2}{1 + x}$

2. $x^2-2x+3-\frac{3}{x-2}$

3. $x^3-3x+x^2-5+\frac{2x-1}{x^2+1}$

96. 2. Addition, Subtraction.

Add $\frac{2}{3}+\frac{5}{3}$ Add $\frac{a}{n}+\frac{b}{n}$

What reason can you give for doing the work the way you have just done it? Does the following reasoning throw any light upon the matter?

$$\frac{2}{3}+\frac{5}{3}=2\times\frac{1}{3}+5\times\frac{1}{3}=(2+5)\times\frac{1}{3}=\frac{2+5}{3}=\frac{7}{3}$$

or in general.

$$\frac{a}{n}+\frac{b}{n}=a\times\frac{1}{n}+b\times\frac{1}{n}=(a+b)\times\frac{1}{n}=\frac{a+b}{n}$$

so also.

$$\frac{a}{n}+\frac{b}{n}-\frac{c}{n}=\frac{a+b-c}{n}$$

Stated in words the conclusion just arrived at is:—

The algebraic sum of several fractions having a common denominator is a fraction whose numerator is the algebraic sum of the numerators of the given fractions and whose denominator is that denominator.

Collect into one fraction, that is, find the algebraic sum.

1. $\frac{7a}{n}+\frac{a}{n}-\frac{4a}{n}$
2. $\frac{3a-b}{c}+\frac{b}{c}-\frac{2a}{c}$
3. $\frac{2a}{9}-\frac{2a-b-c}{9}$
4. $\frac{2R-3r}{R-r}+\frac{2r-R}{R-r}$
5. $\frac{3p-q}{2p-q}+\frac{p-q}{2p-q}$
6. $\frac{x^2-1}{x^2+1}+\frac{x^2+3}{x^2+1}$

97. If the fractions to be added do not have a common denominator they can be reduced to equivalent fractions having a common denominator and then added. Add $\frac{1}{2} + \frac{2}{3}$ what reasons have you for thinking that your work is correct?

In general $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd}$

For the sake of simplicity it is better to use the Lowest Common Multiple of the denominators as the common denominator. We then have the following rule for reducing fractions to a common denominator.

1st. Find the L.C.M. of the denominators of the fractions.

2d. Multiply the terms of each fraction by the quotient arising by dividing the L.C.M. by the denominator of that fraction.

Perform the operations indicated. Reduce results to lowest terms.

7. $\frac{d^2}{2r} - \frac{d^2}{14r}$

8. $\frac{1}{x+3} + \frac{1}{x-2}$

9. $\frac{3}{x-5} - \frac{2}{x+3}$

10. $\frac{a+b}{2} - \frac{2ab}{a+b}$

11. $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}$

12. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$

13. $\frac{1}{m+n} + \frac{1}{m-n}$

14. $2x + \frac{1}{1+x}$

15. $a - \frac{1}{a-1}$

16. $\frac{k}{1-e} - \frac{k}{1+e}$

17. $\frac{7a-b-c}{4} + \frac{2a-b+5c}{6} - \frac{a-5b+7c}{12}$

18. Add $\frac{n-3}{5n}$, $\frac{n^2-9}{10n^2}$, $\frac{8-n^3}{15n^3}$

19. $\frac{2a+1}{a+1}-\frac{3a-1}{a-1}+\frac{a(a+3)}{a^2-1}$

20. $\frac{3x}{4x^2-9}+\frac{7}{2x+3}-\frac{2}{2x-3}$

21. $\frac{4}{x}-\frac{3}{x+1}-\frac{2}{(x+1)^2}$

22. $\frac{2}{(x-3)^3}-\frac{5}{(x-3)^2}+\frac{1}{x-3}$

23. $\frac{1}{a}-\frac{1}{a+b}-\frac{1}{a+2b}$

24. $\frac{2}{x}+\frac{5}{3(2-x)}-\frac{1}{3(2+x)}$

25. $\frac{x+1}{x^2+5x+6}+\frac{x+3}{x^2+3y+2}$

26. $\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}$

27. $\frac{n(n-1)}{2}+\frac{n(n+1)(n-1)}{6}$

28. $n+\frac{n(n-1)}{2}\times 2+\frac{n(n-1)(n-2)}{6}$

Put in factored form.

29. $\frac{n(n-1)\ (n-2)}{2\cdot 3}+\frac{n(n-1)(n-2)(n-3)}{24}$

Put in factored form.

30. $a-\frac{Aa+Bb}{A+B}$

31. $\frac{m}{(x+h)^2}-\frac{m}{x^2}$

32. $\frac{y^2}{x+y}+x-y$

33. $1+x+x^2+\frac{1}{1-x}$

34. $\frac{cd}{(c+d)^2}+\frac{(c-d)^2}{4(c+d)^2}$

35. $\frac{hR}{4(R-r)}-\frac{3}{4}\times\frac{r^2h}{(R^2-r^2)}$

36. $\frac{1}{2g^2-g-1}-\frac{1}{2g^2+g-3}$

37. $\frac{T^2-2a^2T+a^4}{4a^2}-\frac{1}{4}a^2$

In some cases the work is much simplified by a judicious grouping of the terms.

38. $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}$

Suggestion: Add the first two, then the last two, then the two results.

39. $\frac{1}{1-n}+\frac{1}{1+n}+\frac{2}{1+n^2}+\frac{4}{1+n^4}$

40. $\frac{1}{x-3}-\frac{3}{x-1}+\frac{3}{x+1}-\frac{1}{x+3}$

41. $\frac{a}{a-b}+\frac{b}{b-a}$

This example can be worked in a much simpler way.

How do the denominators compare in value?

In accordance with the answer to this question we may write $\frac{a}{a-b}+\frac{b}{-(a-b)}$ which, by the law of signs for division becomes $\frac{a}{a-b}-\frac{b}{a-b}$ from which we get the same result before.

Note that we have changed the fraction $\frac{b}{b-a}$ into an equivalent fraction $-\frac{b}{a-b}$ of a more convenient form.

42. Show that $\frac{a-b}{b-a}=-1$

43. Show that $\frac{x-2}{x-5}=\frac{2-x}{5-x}$

44. Show that $\frac{a}{b}=\frac{-a}{-b}=-\frac{a}{-b}=-\frac{-a}{b}$

From this last exercise it may be seen that in any fraction we may consider three signs,

$$+\frac{-a}{+b}$$

the sign of each term and the sign of the fraction.

If the numerator be multipied by -1 the sign of the fraction is changed.

If the denominator be multiplied by -1 the sign of the fraction is changed.

If both terms are multiplied by -1 the sign of the fraction is unchanged.

It follows that one may make, at pleasure, the following changes of sign without changing the value of the faction.

Change signs of both terms.

Change sign of numerator and sign of fraction.

Change sign of denominator and sign of fraction.

We may make use of the suggestions given above in such examples as the following, where certain denominators differ in sign.

45. Add $\frac{a}{a-b}+\frac{b}{b-a}+\frac{1}{a-b}$

46. $\frac{1}{x-1}-\frac{4}{1-x}-\frac{6}{1+x}$

47. $\frac{2}{x^2-1}-\frac{1}{x-1}-\frac{1}{1-x}+\frac{2}{1-x^2}$

48. $\frac{a}{a-b}+\frac{b}{b-a}-\frac{1}{a-b}$

49. Is $\frac{1}{(x-y)(y-1)}=\frac{1}{(y-x)(y-1)}$?

50. What about $\frac{1}{(x-y)(x-1)}$ and $\frac{1}{(y-x)(1-x)}$

51. Is $\frac{1}{(x-1)^2}=\frac{1}{(1-x)^2}$?

52. Add $\frac{a}{(a-b)(b-c)}+\frac{b}{(b-a)(b-c)}$

53. $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}+\frac{1}{(a-c)(b-a)}$

99. 3. MULTIPLICATION.

$$\frac{2}{5}\times\frac{3}{7}=?$$

How do you find the product? Can you give any reason for finding it in that way?

What do you think of the following course of reasoning? or in general.

$$\frac{2}{3}\times\frac{5}{7}=\frac{\frac{2}{3}\times 5}{7}=\frac{\frac{2\times 5}{3}}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}$$

or in general. $\frac{a}{b}\cdot\frac{c}{d}=\frac{a/b\,.\,c}{d}=\frac{ac/b}{d}=\frac{ac}{bd}$

We can therefore derive a rule for the multiplication of actions:—

The product of two fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.

Before actually performing the multiplications in any given case, it is well to divide out all factors common to the terms. Why can this be done?

Perform the following multiplications reducing results to lowest terms.

1. $\frac{3ax^3}{a+x}$ by $2ax$

2. $\frac{4a-x}{a^2-x^2}$ by $a-x$

3. $\frac{5r^3-2r^2}{r+1}$ by $r-1$

4. $\frac{x^2+3x-1}{x^3-3x}$ by $x-2$

5. $\frac{3a}{4b} \cdot \frac{2a}{5b}$

6. $\frac{3n}{5t} \cdot \frac{10nt}{9n^4}$

7. $\frac{5r^2}{7nt} \cdot \frac{21n^2r}{15t^3}$

8. $\frac{a+b}{2} \cdot \frac{3}{a+b}$

9. $\frac{x^2-y^2}{a} \cdot \frac{b}{x-y}$

10. $\frac{\pi r}{2} \cdot \frac{4r}{3\pi}$

11. $\frac{4K}{\pi r^4} \cdot \frac{1}{2r} \cdot \frac{\pi r^2}{a}$

12. $\frac{R^2-r^2}{t^2} \cdot \frac{t^2-t}{R^2-2rR+r^2}$

13. $\frac{3x}{x+y} \cdot \frac{7x+7y}{9}$

14. $\frac{x^2-y^2}{x^2+y^2} \cdot \frac{x^2+y^2}{x-y}$

15. $\frac{x^2+2x+3}{x-3} \cdot \frac{x^2-2x-3}{x-2}$

16. $\frac{e^4-f^4}{(e-f)^2} \cdot \frac{e-f}{e^2-ef} \cdot \frac{e}{e^2+f^2}$

17. $\frac{p^2+p-2}{p^2-7p} \cdot \frac{p^2-13p+42}{p^2+2p}$

18. $\frac{N^2-8N+15}{N^2-12N+35} \cdot \frac{N^2-15N+56}{N^2-17N+72}$

47. $\frac{2}{x^2-1}-\frac{1}{x-1}-\frac{1}{1-x}+\frac{2}{1-x^2}$

48. $\frac{a}{a-b}+\frac{b}{b-a}-\frac{1}{a-b}$

49. Is $\frac{1}{(x-y)(y-1)}=\frac{1}{(y-x)(y-1)}$?

50. What about $\frac{1}{(x-y)(x-1)}$ and $\frac{1}{(y-x)(1-x)}$

51. Is $\frac{1}{(x-1)^2}=\frac{1}{(1-x)^2}$?

52. Add $\frac{a}{(a-b)(b-c)}+\frac{b}{(b-a)(b-c)}$

53. $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}+\frac{1}{(a-c)(b-a)}$

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Perform the following multiplications reducing results to lowest terms.

1. $\frac{3ax^3}{a+x}$ by $2ax$
2. $\frac{4a-x}{a^2-x^2}$ by $a-x$
3. $\frac{5r^3-2r^2}{r+1}$ by $r-1$
4. $\frac{x^2+3x-1}{x^3-3x}$ by $x-2$
5. $\frac{3a}{4b} \cdot \frac{2a}{5b}$
6. $\frac{3n}{5t} \cdot \frac{10nt}{9n^4}$
7. $\frac{5r^2}{7nt} \cdot \frac{21n^2r}{15t^3}$
8. $\frac{a+b}{2} \cdot \frac{3}{a+b}$
9. $\frac{x^2-y^2}{a} \cdot \frac{b}{x-y}$
10. $\frac{\pi r}{2} \cdot \frac{4r}{3\pi}$
11. $\frac{4K}{\pi r^4} \cdot \frac{1}{2r} \cdot \frac{\pi r^3}{a}$
12. $\frac{R^2-r^2}{t^5} \cdot \frac{t^2-t}{R^2-2rR+r^2}$
13. $\frac{3x}{x+y} \cdot \frac{7x+7y}{9}$
14. $\frac{x^2-y^2}{x^2+y^2} \cdot \frac{x^2+y^2}{x-y}$
15. $\frac{x^2+2x+3}{x-3} \cdot \frac{x^2-2x-3}{x-2}$
16. $\frac{e^4-f^4}{(e-f)^2} \cdot \frac{e-f}{e^2-ef} \cdot \frac{e}{e^2+f^2}$
17. $\frac{p^2+p-2}{p^2-7p} \cdot \frac{p^2-13p+42}{p^2+2p}$
18. $\frac{N^2-8N+15}{N^2-12N+35} \cdot \frac{N^2-15N+56}{N^2-17N+72}$

19. $\frac{1}{2t}\left(\frac{1}{(a-x)^2}-\frac{1}{a^2}\right)$

20. $\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+1}\right)$

21. $\frac{a^3-b^3}{a^3+b^3}\cdot\frac{a+b}{a-b}\cdot\frac{(a^2-ab+b^2)^2}{(a^2+ab+b^2)^2}$

22. $(x^3+\frac{1}{x^3})(x+\frac{1}{x})-(x^2+\frac{1}{x^2})$

23. $(a/x-x/a)(a/x-x/a)$.

24. $(1+x^2/y^2)\frac{x}{y}$

25. $(b+\frac{ab}{b-a})(b-\frac{ab}{a+b})(\frac{a^2-b^2}{b^2+a^2})$

26. $\frac{r^n}{n(n+1)}\cdot\frac{(n-1)n}{r^{n-1}}$.

27. $\frac{3y^{n+2}}{x^{n-1}}\cdot\frac{x^{n+1}}{2y^{n-1}}$

28. $\frac{4J}{\pi r^4 2r}\cdot\frac{\pi r^2}{2}\cdot\frac{4r}{3\pi}$

100. 4. INVOLUTION.

$$\left(\frac{2}{3}\right)^4=\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{2^4}{3^4}$$

or in general $$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$$

That is: A power of a fraction equals that power of the numerator divided by the same power of the denominator.

Find the powers.

1. $\left(\frac{5}{3}\right)^2$, $\left(\frac{1}{2}\right)^3$, $\left(\frac{5}{7}\right)^4$, $\left(\frac{a}{b}\right)^3$

2. $\left(\frac{2a^2}{2b}\right)^2, \left(\frac{2n^3}{5r}\right)^3, \left(\frac{2n^2t}{3s}\right)^5$

3. $\left(\frac{x-y}{x+7}\right)^2, \left(\frac{3ab^2c}{a+b}\right)^3, \left(\frac{x}{y}+\frac{y}{x}\right)^2$

4. $\left(\frac{2(r+1)}{3t^2}\right)^3, \left(\frac{abx^2}{cy}\right)^n$

5. $\left(\frac{R(r+1)}{(r+a)^2}\right)^2, \left(\frac{R(r+1)}{(r+a)}\right)^n$

5. DIVISION.

101. It was found in arithmetic that to divide one fraction by another one inverted the divisor and then multiplied.

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

Let us test this to see if it be true in general. We are to test the truth of the statement:

$$\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \times \frac{n}{m}$$

Multiply both members by m/n

$$\frac{a}{b} \div \frac{m}{n} \times \frac{m}{n} = \frac{a}{b} \times \frac{n}{m} \times \frac{m}{n}$$

whence

$$\frac{a}{b} = \frac{a}{b}$$

which shows the truth of the statement above.

We are therefore safe in saying: The quotient of two fractions is obtained by multiplying the dividend by the reciprocal of the divisor.

Perform the following divisions.

1. Divide $\dfrac{63x^2}{x-a}$ by $9x$

2. Divide $\dfrac{7x^3}{x-a}$ by $x+a$

3. Divide $\dfrac{3x^2-2x^3}{2a(x-1)}$ by x^3-x

4. Divide $\dfrac{21a^2b^3x^5}{x^2-2ax+a^2}$ by $7abx^2(x-a)$

5. $\dfrac{2a}{3c} \div \dfrac{3c}{4a}$

6. $\dfrac{2axy}{5b^2} \div \dfrac{6a^2y^2}{10bx^2}$

7. $\dfrac{a^2-x^2}{(a-x)^2} \div \dfrac{a^3+x^3}{a^3-y^3}$

8. $\dfrac{x^2+7x+12}{x^2-4x+4} \div \dfrac{x+4}{x-2}$

9. $\dfrac{a+b}{a^2-ab} \div \dfrac{ab+a^2}{ab-b^2}$

10. $\dfrac{3n^2x}{4a^2b} \div \left(-\dfrac{3x}{2a}\right)$

11. $\dfrac{2(a-1)}{3(x+1)} \div \dfrac{4(1-a)}{5(x-1)}$

12. $\dfrac{x^2+3x}{x+4} \div \dfrac{x+3}{x^3+4y^2}$

13. $\dfrac{x^n}{x(n+1)} \div \dfrac{n(x^{n-1})}{n^2-1}$

14. $\dfrac{5y^2}{7a^3} \div \dfrac{21c^3}{4ax} \div \dfrac{35c^2y}{7a^3x}$

15. $\dfrac{5a(3x-7y)+6b(7y-3x)}{5a-6b)\ (2y+3y)} \div \dfrac{3x-7y}{4x^2-9y^2}$

16. $$\frac{n^2 - n - 20}{n^2 - 25} \div \frac{n + 1}{n^2 - 25} \div \frac{n^2 + 2n - 8}{n^2 - n - 2}$$

17. $$\frac{t^2 - 11t + 30}{t^2 - 6t + 9} \div \frac{t^2 - 5}{t^2 - 3t} \times \frac{t^2 - 9}{t^2 - 36}$$

18. $$\frac{1 + x - x^2 - x^3}{1 - y^2} \div \frac{x^2 - 1}{y^2 - 1}$$

20. $$\frac{6x^{2n} - 24}{x^{2n+3} + 6x^{n+3} + 9x^3} \div \frac{3x^n + 6}{x^3}$$

21. $$\left(\frac{a + b}{a - b} - \frac{a^2 + b^2}{a^2 - b^2}\right) \div \left(\frac{a - b}{a + b} - \frac{a^3 - b^3}{a^3 + b^3}\right)$$

22. $$1 - \frac{a + b}{a - b}\left(\frac{a}{a + b} - \frac{a - b}{a} + \frac{a - b}{a + b}\right)$$

6. Complex Fraction.

102. When the division of two expressions, either or both of which contain fractions, is expressed in the form of a fraction it is called a complex fraction.

Thus $$\frac{1 + \frac{1}{x}}{1 - x}, \quad \frac{2 + \frac{1}{a}}{\frac{1}{a^2} - 1}, \quad \frac{\frac{x^2 + 3x}{x + 4}}{\frac{x + 3}{x^3 + 4x}}$$

There are two methods that may be employed in reducing such fractions to simpler forms.

Remembering that a fraction is but an indicated division we may put

$$\frac{\frac{x^2 + 3x}{x + 4}}{\frac{x + 3}{x^3 + 4x^2}} = \frac{x^2 + 3x}{x + 4} \times \frac{x^3 + 4x^2}{x + 3} = x^3$$

In many cases, however, it is much simpler and better to make use of the third law of fractions and multiply each term of the complex fraction by the Least Common Denominator of the minor fractions.

$$\frac{1+\frac{1}{x}}{1-x}=\frac{(1+\frac{1}{x})x}{(1-x)x}=\frac{x-1}{(1-x)x}=-\frac{1}{x}$$

so also
$$\frac{2+\frac{1}{a}}{\frac{1}{a^2}-1}=\frac{2a^2+a}{1-a^2}$$

Reduce to simple fractions.

1. $\dfrac{1/x-3}{3+2/x}$

2. $\dfrac{a/6-a/8}{24}$

3. $\dfrac{2/x+y/2+x}{3/x+y/3+y}$

4. $\dfrac{a/b-b/a}{a/b+b/a}$

5. $\dfrac{1-\frac{x-1}{2}-x}{1-\frac{1-x}{2}-x}$

6. $\dfrac{\frac{a+2}{a}+\frac{6}{a-3}}{\frac{a^2-a+2}{a-3}+1}$

7. $\dfrac{\frac{2}{3}-\frac{5}{6}}{1+\frac{2}{3}\times\frac{5}{6}}$

8. $\dfrac{\frac{2h}{1+h^2}+h}{1-\frac{2h^2}{1-h^2}}$

9. $x-\dfrac{\left(1+\frac{x^2}{y^2}\right)\frac{x}{y}}{\frac{x^2+y^2}{y^2}}$

10. $y-\dfrac{1+\frac{a^2-y^2}{y^2}}{\frac{a^2}{y^2}}$

11. $$\frac{1}{1+\frac{q+1}{n-q}}$$

12. $$\frac{1}{3(x-1)}+\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}$$

13. If a be added to the numerator of a fraction the result will be b, if a be added to the denominator of a fraction the result will be c.

What is the fraction? Solve and check.

14. If 1 be added to the numerator of a fraction the result will be 1/a, if 1 be added to the denominator of a fraction the result will be 1/b, what is the fraction?

Solve and check.

15. $$\frac{\frac{1-a}{1+a}-1}{\frac{1+a}{1-a}-1}$$

16. $$\frac{\frac{v-2}{v+2}-\frac{v+2}{v-2}}{\frac{v-2}{v+2}+\frac{v+2}{v-2}}$$

17. $$\frac{(1/x+1/y)^2-1}{(1/x+1/y)-1}$$

18. $$\frac{1}{2}\left(\frac{\frac{a}{e}-ae}{1+e}+\frac{\frac{a}{e}-ae}{1-e}\right)$$

19. $$\frac{1}{1-\frac{1}{1+1/a}}$$

20. $$\frac{1}{1-\frac{1+x^2}{x-\frac{1}{1-x}}}$$

21. $$\frac{(1-1/x)^2+(1+1/x)^2}{(1+1/x)^2-(1-1/x)^2}$$

22. $$\frac{a^2-1}{n^2+n}\times\left(1-\frac{1}{1-1/n}\right)\div\frac{1-a^2}{1+n-n^2-n^3}$$

7. RATIO.

103. The ratio of one number to another is the quotient arising from dividing the first by the second. The two numbers are called the terms of the ratio and are often appropriately distinguished as the antecedent and the consequent.

The ratio of 12 to 4 is 3

This may be written in several ways:

$$\frac{12}{4}=3\ ,\ 12/4=3\ ,\ 12\div4=3\ ,\ 12:4=3$$

The fractional notation is rather to be preferred.

The ratio of a to b is, say, r

$$\frac{a}{b}=r$$

Note that a ratio is a number; it is a quotient.

Are all fractions ratios?
Are all ratios fractions?

1. Express the ratio of 4 to 6.

2. Express the ratio of 3 inches to 2 feet.

3. What is the ratio of the length to the width of a rectangle of dimensions, 3 feet by 7 feet?

4. Is the ratio of 3 feet to 14 feet, 1/2 feet?

5. Two rectangles have the dimensions 6×15 and 10×18: what is the ratio of the areas?

6. The dimensions of one box are, 4, 6, 10, the dimensions of another are, 6, 8, 12; what is the ratio of the volume of the first to the volume of the second: of the second to the first?

7. Express the ratio of 3 to 12 in the form of a decimal.

8. Find the ratio of 2 to 7, correct to their decimal places.

9. Measure the length of this piece of paper in both English and metric standards. And find the number of centimeters in one inch. What is the ratio of an inch to a centimeter?

10. What is the ratio of a mile to a kilometer? Of a kilometer to a mile?

11. In 1900 the population of the United States was 76,304,799, while in 1890 it was 62,622,250; find the ratio of the population in 1900 to that in 1890 correct to .01.

12. The density of population is the ratio of the population to the area, that is, it is the number of people to the square mile. Find the density of population correct to one decimal place, for the following countries, and show the results graphically by means of a series of straight lines of the proper length.

COUNTRY	AREA	POPULATION
United States	3,734,866	76,304,799
Great Britian	121,115	40,599,954
France	204,092	38,518,075
Germany	208,830	52,279,915
China	3,278,401	402,680,000
India	1,560,160	287,223,431

13. In a lot of seven dozen eggs the ratio of good to bad is 2 to 3; how many good eggs are there in the lot?

14. Find two numbers in the ratio of 7 to 8 whose sum is 90.

15. Which is the greater ratio, 2/3 or 3/4?

16. Find the ratio of br^3 to ar^2.

17. What is the ratio of a/b to c/d?

18. What is the ratio of 1/x to 1/y?

19. What is the ratio of $\frac{a}{x-a}$ to $\frac{3a}{(x-a)^2}$?

20. Find the ratio of $\frac{x^{n-1}}{n^2-1}$ to $\frac{x^{n-2}}{(n-1)^2}$?

21. A rectangle is 3 by 5: squares are constructed on one end and on one side: would the areas of the squares have the same ratio as the sides of the rectangle?

22. Find the ratio of $\frac{n(n-1)}{2}$ to $\frac{n(n-1)\ (n-2)}{2 . 3}$

23. From $1/f = 1/v - 1/u$ find an expression for the ratio v/u; also for v/f.

24. What is the ratio of one side of a square to the other side?

25. What is the ratio of the diagonal of a square to its side?

26. What is the ratio of the circumference of a circle to its diameter?

27. What effect does a change in the diameter have upon this ratio?

28. We say then that the circumference of a circle varies as the diameter.

Whenever the ratio of two variable magnitudes is constant we say that one varies as the other.

It follows that the one is some constant number times the other. Why?

The circumference of a circle varies as the radius; what is the constant in this case?

29. The area of a circle is given by the formula $A = \pi r^2$; the area varies as — ?

Express the following statements in algebraic terms.

30. In the case of a body falling from rest it has been found that the velocity at any time varies as the time? The space passed thru varies as the square of the time. Resistance to motion in the air varies as the square of the velocity.

31. The volume of a box varies as the product of its three dimensions. What is the constant in this case?

32. What algebraic operations can be performed on the terms of a ratio without changing the value of the ratio?

33. What can you say about the relation of the ratios 2 : 3 and 12 : 18?

34. Consider two circles of circumferences C, c, and of radii R, r; since the ratio C/R is constant we have

$$C/R = c/r$$

What is the ratio of the two circumferences?

When four numbers are so related that the ratio of two of them is equal to the ratio of the other two, the numbers are said to be proportional to each other.

A proportion is thus seen to be an equation. It can be solved like any equation.

35. What is the ratio of the circumferences of two circles the diameter of one being four inches and the other two inches?

36. What is the ratio of the areas of the same circles?

37. Find the ratio of the areas of the United States and Australia. Represent these areas by means of two circles.

38. By means of circles show the comparative areas of Illinois, Massachusetts, Texas.

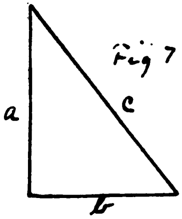


Fig 7

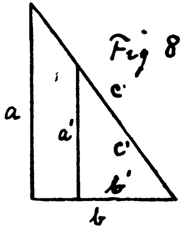


Fig 8

39. One a sheet of co-ordinate paper draw a triangle like Fig. 7, of any dimensions you choose; let the vertices be at the crossing of the half inch lines. Call the sides a, b, c.

Now draw another one on the same figure to make it look like Fig. 8, calling the sides of the smaller triangle a′, b′, c′.

Now find the ratios $\frac{a}{a'}$, $\frac{b}{b'}$, $\frac{c}{c'}$

What do you find true of these ratios?

Such triangles are said to be similar, and a and a′, b and b′, c and c′, are called corresponding sides.

Suppose now in two triangles, $a = 20$ feet; $a' = 12$ feet, and $b' = 8$ feet, what will be the length of b?

Let $a' = 7$, $b' = 3$, $b = 20$, $a = ?$

This fact about similar triangles may be used in the finding of heights and distances.

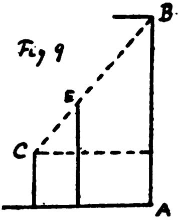


Fig 9

40. Let us find the height of this room.

Let AB be the wall.

Let C be my eye.

Now place this eight foot pole so that its top is in line with my eye and the top of the wall say at DE.

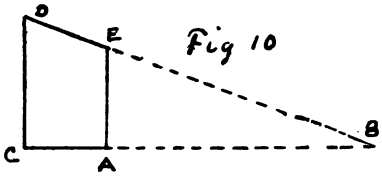
What lines shall I measure in order to compute the length of AB?

Make the measurements and solve the resulting equation. Test the accuracy of our work by actually measuring the height of the room directly.

41. Find the height of the end of the electric light pole in front of this building.

42. Find the height of one of the poplars at the rear of his building.

43. Find the height of the corner of the top of the tower on Bradley Hall.



44. The same principle can be applied to the finding of horizontal distances.

Find the distance of St. Marks church from the center of the circle in front of the Institute.

Let B be St. Marks and A the center of the circle, set a pole at A and measure A C in line with B. 200 feet. Now make B A E a right angle and measure A E. 100 feet say, Put a pole at E. Make B C D a right angle, and line in D measure C D. Write the equation needed to find A B and solve How will you lay off the right angles required? Try the same plan on some problem of your own choosing and hand your report in.

45. Find the height of an object of your own choosing by means of its shadow.

CHAPTER 8.

ROOTS, EXPONENTS, RADICALS.

104. The square root of a number is one of its two equal factors.

$3^2 = 9$ hence the square root of 9 is 3 which statement is usually written $\sqrt{9} = 3$.

But we have seen that $-3 \times -3 = 9$ also, therefore it follows that -3 is also a square root of 9; so that

$$\sqrt{9} = 3 \text{ or } -3$$

In fact every number has two square roots which differ only in sign: the positive root is called the principal root.

For in general

$$a \times a = a^2 \text{ and } -a \times -a = a^2$$
$$\text{therefore} \quad \sqrt{a^2} = +a \text{ or } -a$$

In our discussion we will notice the principal root only, unless it is otherwise noted.

To obtain the square root of a number we have but to separate it into two equal factors: one of these factors is the square root

$$\sqrt{a^6} = ? \quad a^6 = a^3 \,.\, a^3, \text{ therefore } \sqrt{a^6} = a^3$$

The result may be best gotten by dividing the exponent by 2. In getting the square of a number we multiplied its exponent by 2, so here in getting the square root of a number we divide by 2. Stated in general this is

$$\sqrt{a^n} = a^{\frac{n}{2}}$$

105. Find the square root of $a^6 b^8$

$$a^6b^8 = a^3a^3 \,.\, a^4b^4$$

$$\sqrt{a^6b^8} = a^3b^4$$

$$\text{i. e. } \sqrt{a^6b^8} = \sqrt{a^6}\sqrt{b^8} = a^3 b^4$$

stated in general this is $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

The square root of a product equals the product of the square roots of the factors.

106. Find the square roots of

1. $25a^6b^4$
2. $16x^2y^8$
3. $625n^6$
4. $49h^4$
5. $225n^2t^4$
6. $81R^2$
7. $196t^{10}$
8. $576x^4y^6z^2$
9. $(x-q)^2$
10. $x^2(x+2)^6$
11. $(x+y)^4$
12. $49(x-1)^6$
13. $9n^4(n+1)^2$
14. $25x^{2n}y^4$
15. $64a^2x^{8n}y^{8n}$

107. So also the cube root of a number is one of the three equal factors of the number.

The cube root of 27 is 3, or in algebraic language $\sqrt[3]{27} = 3$.

As in the case of square root so here in finding the cube root we may separate the number into three equal factors; one of these is the cube root.

$$8a^6 b^9 = 2a^2 b^3 \,.\, 2a^2 b^3 \,.\, 2a^2 b^3$$

$$\sqrt[3]{8a^6 b^9} = 2a^2 b^3$$

Here also a better way is to divide the exponents of the factors by 3.

that is in general $\sqrt[3]{a^n} = a^{n+3}$

Find the cube roots of

1. $64a^3 b^6$
2. $8a^3x^9$
3. $125c^6n^{12}$
4. $27a^3 b^6x^{15}y^9$
5. $(2x+7)^6$
6. $343x^{3n}$
7. $\frac{1}{8}n^3x^9y^{21}$
8. $a^3x^9y^{3t}$
9. $8(x-1)^3$

108. The fourth root is one of the four equal factors of a number.

And so in general the nth root of a number is one of the n equal factors of a number.

The nth root can be found by the use of the same method:—

$$\sqrt[n]{a^m} = a^{m \div n}$$

$$\text{also } \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

1. Find the 4th root of $625x^4y^{16}$, of $16a^4x^8$
2. Find the 5th root of $32x^5y^{15}$
3. Find the nth root of $a^{2n}b^{3n}$
4. Find the pth root of x^py^{4p}

109. In the cases we have considered, the exponents have all been exactly divisible. We will apply the same method to any case whatsoever.

We have seen that $\sqrt{a^6} = a^{\frac{6}{2}} = a^3$

$a^{\frac{6}{2}}$ means the same as $\sqrt{a^6}$ and a^3 is the square root of a^6 so also $\sqrt{a^3} = a^{\frac{3}{2}} = {}^{1\frac{1}{2}}$

$a^{\frac{3}{2}}$ means the same as $\sqrt{a^3}$ and will call $a^{\frac{3}{2}}$ or $a^{1\frac{1}{2}}$ the square root of a^3

We have here a new sort of exponent, a fraction, a kind we have not encountered before.

The numerator of the fraction indicates a power and the denominator, a root.

If $a^{\frac{3}{2}}$ is the square root of a^3 then $a^{\frac{3}{2}}.a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^3$ which is in perfect accord with the law of exponents for multiplication.

Find the square root of $9a^3b^4$

$$\sqrt{9a^3b^4} = 3a^{\frac{3}{2}}b^2 = 3ab^2a^{\frac{1}{2}}$$

Find the roots of the following monominals:

1. $\sqrt{25xy^4}$
2. $\sqrt{4a^2x^8}$
3. $\sqrt{100x^2y^5}$
4. $\sqrt{169n^7t^8}$
5. $\sqrt[3]{8a^3b^6}$
6. $\sqrt[3]{27n^6t^{10}}$
7. $\sqrt[3]{R^3T}$
8. $\sqrt[4]{16R^8t^5}$
9. $\sqrt[n]{a^{2n}b^n}$
10. $\sqrt[n]{x^{3n}y^2}$

110. Thus far we have been considering the finding of the roots of monomials. When, however, it comes to the finding of the square root (or any root, for that matter) of a polynomial we find a more difficult problem. Various ways have been devised. In this course we will consider two methods of finding the square root of a polynomial.

111. If the polynomial is easily factored the method used for nominals can often be applied. Separate into equal factors.

Find the square roots of

1. $x^2 + 2xy + y^2$
2. $a^2 + 6a + 9$
3. $4x^2 - 12xy + 9y^2$
4. $25a^2 + 49b^2 + 70ab$

Find the cube roots of

5. $x^3 - 3x^2 + 3x - 1$
6. $x^3 + 6x^2 + 12x + 8$

But this method is applicable in only a few cases and it is difficult to determine when it can be used.

We should have a more general method that can be used in all cases. You are already acquainted with it from your study of arithmetic.

112.
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$\text{so that } a + b = \sqrt{a^2 + 2ab + b^2}$$

Note that the first term of the square root is the square root of the first term of the expression whose square root is required. If the square of this be subtracted there remains $2ab + b^2$.

The second term of the root may be found by dividing this remainder by twice the first term of the root: it is also seen that the remainder, $2ab + b^2 = (2a + b)b$, is the second term of the root times twice the first term plus the second term.

Again

$$(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$
$$= a^2 + (2a + b)\,b + (2a + 2b + c)c$$

If we subtract in turn, the a^2 and the $(2a + b)b$ we have the remainder $2ac + 2bc + c^2$, the third term can be found by dividing this remainder by twice the first term, and the remainder is seen to be of the same form as before

$$(2a + 2b + c)c$$

This suggests to us, then, a general method which we will illustrate more fully by an example.

Find the square root of

$$4x^4 + 12x^3 + 17x^2 + 12x + 4$$

We arrange the work thus:

$$\begin{array}{lll}
 & & a + b + c \\
 & N = 4x^4 + 12x^3 + 17x^2 + 12x + 4 & (2x^2 + 3x + 2 \\
 & 4x^4 & \\
2a = 4x^2 & 12x^3 + 17x^2 + 12x + 4 \; (= N - a^2) & \\
2a + b = 4x^2 + 3x & 12x^3 + 9x^2 \qquad (=(2a + b)b) & \\
2(a + b) = 4x^2 + 6x & \quad + 8x^2 + 12x + 4 & \\
2(a + b) + c = 4x^2 + 6x + 2 & \quad 8x^2 + 12x + 4 &
\end{array}$$

You will notice that the method is the same you have used in arithmetic and you now can see a meaning in what you did then.

Find the square root of

1. 53361
2. $x^4 + 2x^3 + 3x^2 + 2x + 1$
3. $4x^4 + 12x^3 + 13x^2 + 6x + 1$
4. $n^6 - 2n^5 + 3n^4 - 4n^3 + 3n^2 - 2n + 1$
5. $4y^4 - 12y^3 - 7y^2 + 24y + 16$
6. 529
7. 3721
8. 18729

113. We have defined the square root of a number as one of its two equal factors.

But 5 cannot be separated into two equal factors and it would seem, then, that 5 can have no square root. When the method of the last article applied to 5, as you have seen before, the work never terminates, and the longer it continued the closer is the approximation to a number between 2 and 3. This number whose exact value we cannot find is the square root of 5. We may get as close an approximation to it as we choose.

This sort of a square root is a kind of number very different from the other kinds of number we have been using; integers, negatives, and fractions.

A root whose exact value cannot be expressed in decimals is called an irrational number.

9. Make a table of the square roots of the natural numbers from 1 to 10 showing three decimal places. Be sure to preserve this table in your books, for you will often have need to refer to it.

10. Find the square roots of 18, 32, 50, and 98 in two ways.

11. Can you find the square root of 26 in two ways? Why?

114. A similar method for extracting the cube root of a number can be developed. It is, however, of much less importance to us, for our present work. In fact we would use it only in finding the cube roots of a few numbers, in some problems in Chapter 11. And before we come to them we will have discovered a much better way of getting any kind of a root of a number, be it a square root, or a cube root or a fifth root. If you are consumed with curiosity to develop a rule for finding the cube root you can find it in any of the algebras in the library.

115. As we have seen there are two ways of indicating a root:—

$$a^1 \text{ and } \sqrt{a}$$

The first may be called the exponential form, and the second the radical form. Either form, may be used at pleasure. For some purposes the one form is to be preferred and for other purposes the other form.

Expressions containing fractional exponents or radical forms are called irrational expressions. They indicate that a required root can be found only approximately. They are constantly appearing in mathematical work. The exponential notation is exceedingly useful, and a mastery of its correct use is imperative if you wish to know anything about algebra and if you wish to use algebra in the pursuit of other knowledge.

We have seen that a positive integral exponent indicates that a number with which it is affected is to be used that many times as a factor.

In the case of a fractional exponent the numerator indicates a power and the denominator indicates a root.

The fraction is just as much an exponent as is an integers, though in one way it has a somewhat different interpretation.

Interpret the meaning of the exponents in the following and find the numerical values.

1. 3^2 , 2^3 , 2^5 , 5^2 , 7^3
2. 4½ , 8½ , $8^{\frac{2}{3}}$, 4 , $16^{\frac{3}{4}}$
3. 3½ , 2½ , 2 , $4^{.5}$, $16^{.75}$

116. Thruout our work so far we have made constant use of certain laws or rules of reckoning with exponents.

1. $a^m \,.\, a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab^n = a^n b^n$

We discovered these laws in connection with positive integral exponents. They are, however, perfectly general and may be applied to any case whatever, let the exponents be what they may, positive, negative, integral, fractional.

The results can be more or less easily interpreted and will be found to be perfectly consistent with the results found by any other mode of reckoning.

If you continue the study of algebra very far you will, no doubt become interested in investigating the general nature of these laws. It is a very pretty piece of work but does not especially interest us now. We are interested in the fact that these laws are universal in their application.

This makes reckoning with irrational expressions very simple. As long as we keep to the exponential form no new difficulty arises.

For instance

Addition and subtraction	$2ax^{\frac{1}{2}} + 3ax^{\frac{1}{2}} - 4ax^{\frac{1}{2}} = ax^{\frac{1}{2}}$
Multiplication	$2a^{\frac{1}{2}} \,.\, 3a^{\frac{1}{3}} = 6a^{\frac{1}{2}+\frac{1}{3}} = 6a^{\frac{5}{6}}$
Division	$7a^{\frac{2}{3}} \div 3a^{\frac{1}{3}} = \frac{7}{3}\, a^{\frac{1}{3}}$
Involution	$(2ax^{\frac{1}{2}}y^{\frac{1}{3}})^2 = 4a^2x^{\frac{2}{3}}$
Evolution	$(9a^4x^{\frac{1}{2}}y^{\frac{1}{3}})^{\frac{1}{2}} = 3a^2x^{\frac{1}{4}}y^{\frac{1}{6}}$

117. Perfcrm the operations indicated and interpret results:

1. $27^{\frac{1}{3}} \times 27^{\frac{2}{3}}$
2. $32^{\frac{1}{5}} \times 32^{\frac{3}{5}}$
3. $a^{\frac{2}{3}} \,.\, a^{\frac{1}{6}}$
4. $3b^{\frac{3}{4}} \,.\, b^{\frac{7}{4}}$
5. $5n^{\frac{1}{2}}p^{\frac{2}{3}} \,.\, n^{\frac{3}{2}}p^{\frac{1}{3}}$
6. $6n^{\frac{1}{2}} \div 5n^{\frac{1}{3}}$
7. $27x^2y^3 \div 3xy^{\frac{3}{2}}$
8. $3a^{\frac{1}{2}} \,.\, 4a^{\frac{1}{3}}$

9. $7(a^{\frac{2}{3}})^{\frac{3}{2}}$

10. $(32b^{\frac{5}{6}}$

11. $\frac{b^{\frac{1}{2}}a^{\frac{2}{3}}}{a^{\frac{1}{6}}b^{\frac{3}{4}}} \times \frac{b^3a^5}{a^2b^{\frac{1}{2}}}$

12. $(a^2 + 2ab + b_1)^{\frac{1}{2}}$

13. $(\ (a + b)^{\frac{2}{3}}\ )^2$

14. $(r^2 + s^2 - 2rs)^{\frac{3}{2}}$

15. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})\ \ (x^{\frac{1}{2}} - y^{\frac{1}{2}})$

16. $(n^{\frac{1}{3}} + t^{\frac{1}{3}})\ \ (n^{\frac{2}{3}} - n^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}})$

17. $(k^{\frac{1}{3}} + p^{\frac{1}{3}})\ \ (k^{\frac{1}{3}} - p^{\frac{1}{3}})$

18. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$

19. Divide $(a^2 + x^2)^{\frac{3}{2}} - 3x^2(a^2 + x^2)^{\frac{1}{2}}$ by $(a^2 + x^2)^{\frac{1}{2}}$

20. $(\ (x - 2)^{\frac{2}{3}})^{\frac{3}{4}}$

21. Simplify $\dfrac{\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}}{(1 + \dfrac{x^2}{a^2 - x^2})^{\frac{3}{2}}}$

22. $2x^2 + 3x^{\frac{1}{2}} + 2x^{\frac{1}{3}} - x^2 - x^{\frac{1}{3}} - 4x^{\frac{1}{2}}$

23. $3a^{\frac{1}{2}} + 7b^{\frac{1}{2}} + 6a^{\frac{3}{4}} - 3b^{\frac{4}{5}}$

24. Add $2a^{\frac{1}{2}} + 4a^{\frac{1}{3}} + 5a^{\frac{2}{3}} 4a^{\frac{1}{2}} - 5a^{\frac{1}{3}} - 6a^{\frac{2}{3}}, - 6a^{\frac{1}{2}} - 2a^{\frac{1}{3}} - 3a^{\frac{2}{3}}$

25. $75^{\frac{1}{2}} + 147^{\frac{1}{2}} - 243^{\frac{1}{2}}$

26. $200^{\frac{1}{2}} + 18^{\frac{1}{2}} - 72^{\frac{1}{2}} - 98^{\frac{1}{2}}$

27. $2^{\frac{1}{2}}\ .\ 7^{\frac{1}{2}}\ .\ a^{\frac{1}{2}}\ .\ b^{\frac{1}{2}}$

28. $(3a)^{\frac{1}{3}}\ .\ (2ab)^{\frac{1}{3}}$

29. $5^{\frac{1}{2}}.(\ \frac{1}{5}\ )^{\frac{1}{2}}$

30. $2^{\frac{1}{2}}.2^{\frac{1}{3}}\ ,\ 2^{\frac{1}{2}}.3^{\frac{1}{3}},\ a^{\frac{1}{2}}b^{\frac{1}{3}}$

31. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^3$

32. $(a^{\frac{1}{2}}b^{\frac{3}{4}}x^{\frac{2}{3}})^{\frac{2}{3}}$

118. As we have seen the fractional notation will enable one to easily perform any of the operations with irrational numbers. But in your work in Geometry and Trigonometry you will find the radical form of the irrational more commonly used. It is exceedingly important then to know how to use radicals. (For the sake of brevity we called irrationals expressed in the radical form radicals.)

If in any case there is a question as to what should be done, one has but to write the expression in the exponential form and the perplexity disappears.

119. In the form $\sqrt[n]{3a^2}$

$\sqrt[n]{\ }$ is called the radical sign

n is called the order of the radical

$3a^2$ is called the radicand

When n is 2 it is omitted

We will proceed to show how radical expressions are dealt with, confining ourselves largely to radical expressions of the order two, as they are of the greatest importance. The method of treatment can be easily extended to include others.

1. REDUCTION OF RADICALS.

120. Consider the radical $\sqrt{18a^4b^5}$. Perform the operation indicated and get.

$$\sqrt{18a^4b^5} = \sqrt{9a^4b^4 \, . \, 2b} = 3a^2b^2\sqrt{2b}$$

We have decomposed the radicand into factors, one a perfect square and the other not. The square factor is called a rational factor because its root can be taken.

It is thus seen that a rational factor may be removed from under the radical if its root be taken.

The reverse is equally true, an outside factor may be introduced under the radical sign if it be raised to the power indicated by the order of the radical.

$$2a\sqrt{3ba} = \sqrt{4a^2 \, . \, 3ba} = \sqrt{12a^3b}$$

The form $2a\sqrt{3ba}$ is said to be simpler than the form $\sqrt{12a^3b}$. There is no rational factor under the radical sign.

You may not at first see why it is called simpler. The next exercise may throw light upon the matter.

Find the values of $\sqrt{75}$, $\sqrt{147}$, $\sqrt{243}$.

It will often be found that a problem involving radicals is much simpified if the radicals are reduced to their simplest forms.

$$\sqrt[3]{16a^4} = \sqrt[3]{8a^3 . 2a} = 2a\sqrt[3]{2a}$$

Simplify the following radicals.

1. $\sqrt{20}$, $\sqrt{28}$, $\sqrt{63}$, $\sqrt{125}$, $\sqrt[3]{24}$
2. $\sqrt{8a^7}$, $\sqrt{6a^3}$, $\sqrt{75x^3y^7}$
3. $\sqrt{45n^{13}n^5}$, $\sqrt{250a^{15}b^{11}c^3}$
4. $\sqrt[3]{32}$, $\sqrt[3]{8a^2c^3}$, $\sqrt[3]{16a^5b^3}$
5. $\sqrt[5]{64a^7b^{10}c^4}$, $\sqrt{(x-1)^3}$
6. $\sqrt{(x-y)^3(x+y)}$, $\sqrt{27(x+y)^5}$
7. $x\sqrt{x^3-x^2}$, $\sqrt{(a^3-2a^2b+ab^2)}$
8. $\sqrt{36a^3-72a^2b+b^2a}$, $\sqrt[3]{a^3b^2(b+bc)}$
9. $\sqrt{x}$, $\sqrt{x^{4n+3}}$, $\sqrt{x^{4n+1}}$, $\sqrt{x^{3n}}$
10. $\sqrt{x^{2n+3}}$, $\sqrt{x^{3n+5}}$
11. $\sqrt{3/4}$, $\sqrt{5/16}$, $\sqrt{24/49}$, $\sqrt{3a^2/25}$
12. $\sqrt{\frac{25a^7}{36x^3}}$, $\sqrt{\frac{6xy}{81a^2b^4}}$, $\sqrt{\frac{1ny^2}{9n^2r^4}}$

A radical having no outside factors is called an entire radical.

Reduce to entire radicals.

1. $3\sqrt{2a}$, $4\sqrt{5a}$, $2a\sqrt{5a}$, $3a^2\sqrt{2a}$, $x^3\sqrt{xy}$
2. $3a^3\sqrt{a^3b^2}$, $x^3y^2\sqrt{x^2y^3}$, $5a^4\sqrt{2a^3 \quad ^2}$
3. $(x+6)\sqrt{x-6}$, $\frac{a}{b}\sqrt{bc}$, $r\sqrt{rh}$

4. $\frac{7}{5}\sqrt{\frac{5}{7}}$, $(a-b)\sqrt{a-b}$

5. $(x-y)\sqrt{\frac{x-y}{x+y}}$

121. Compute the value of $\sqrt{\frac{2}{3}}$ to three decimal places.

In how many ways can you make the computation?

I wonder if we cannot find a better way than any you have given. That fraction in the radicand is a sort of a nuisance. It may be that we can get rid of it. If its denominator were a square we could remove it.

The denominator can be made a square thus:—

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{2\times3}{3\times3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6}$$

As we have already computed $\sqrt{6}$, the value of $\sqrt{\frac{2}{3}}$ is now easily computed.

$\frac{1}{3}\sqrt{6}$ is said to be simpler form than $\sqrt{\frac{2}{3}}$

The reason is obvious.

Find $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{2}{75}}$

122. We may now state that a radical is in its simplest form when:—

1) The radicand contains no rational factor.

2) The radicand contains no fraction.

A fraction may always be removed from the radicand by causing its denominator to become rational, and then removing the rational factor.

The denominator can always be made rational by multiplying both numerator and denominator by a properly chosen multiplier called a rationalizing factor.

Simplify the following radicals.

1. $\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{3}{8}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{a}{5}}$

2. $\sqrt{\frac{2ab^2}{3xy^2}}, \sqrt{\frac{5x^3y^7}{6ab^3}}, \sqrt{\frac{11t^2m^5}{4a^3b^5c}}$.

3. $\sqrt[3]{\frac{2}{3}}, \sqrt[3]{\frac{4}{9}}, \sqrt[3]{\frac{3}{4}}, \sqrt[3]{\frac{a}{b}}, \sqrt[3]{\frac{ab^2}{c}}$

4. $x\sqrt{\frac{a}{xy}}, y\sqrt[6]{\frac{a}{xy}}, \sqrt[3]{\frac{2a^2}{3x}}$

5. $\sqrt{\frac{a^2(1+c)^3}{y^3(1-x)^2}}, \sqrt{\frac{8a^{2n}c^{3n}}{b^r}}, \sqrt{\frac{x+y}{x-y}}$

6. $\sqrt{\frac{(x+y)^3}{(x-y)}}, \sqrt{\frac{3a^3}{7b^5}}, \sqrt{\frac{2a(a-b)^5}{(a+b)^3}}$

7. $\sqrt{\frac{0\,.\,15}{5\,.\,4}}, \sqrt{\frac{n^9r^{11}p^1}{2n^2p^5q^7}}$

Reduce to entire radicals.

8. $\frac{a}{b}\sqrt{\frac{bc}{a}}, \frac{1}{3}\sqrt{3}, \frac{3}{2}\sqrt{\frac{2a}{3}}$

9. $3\sqrt[3]{\frac{2a^2}{3}}, \frac{3x}{2}\sqrt[3]{\frac{5x^3}{3}}, \frac{2}{3}\sqrt[3]{\frac{9}{4}}$

10. $\frac{2}{5}a^2c\sqrt{\frac{5b}{2a^2c}}, \frac{n+1}{n-1}\sqrt{\frac{n-1}{n+1}}$

2. ADDITION OF RADICALS.

123. Let it be required to find the numerical value of $\sqrt{162}-\sqrt{98}$.

This can be found in two ways. What are they?

Add $\sqrt{a^3b}+\sqrt{ab^3}-\sqrt{4ab^3}$

So, in general, radicals can be added into one radical if they are of the same order and have the same radicand.

Such radicals are called like radicals. The radicals to be added should always be reduced to their simplest forms so as to reveal the radicals that are alike.

Show why the statement given above in regard to the addition of radicals is true.

Simplify the following expressions and where only numerical values are given express the results in decimals.

1. $2\sqrt{2} - 3\sqrt{3} + 5\sqrt{3} - 6\sqrt{2} + 3\sqrt{3} - 7\sqrt{2}$
2. $3\sqrt{2} - 2\sqrt{8} + 3\sqrt{4} - 3\sqrt{2} - 2\sqrt{16} + 5\sqrt{18}$
3. $a\sqrt{b} - \sqrt{4a^2b} + \sqrt{b^2a} - 3\sqrt{b^2a} + 2\sqrt{ba^2}$
4. $\sqrt{18} - \sqrt{2} + \sqrt{32} - \sqrt{125} + \sqrt{50} - \sqrt{98}$
5. $\sqrt{20} + \sqrt{125} + \sqrt{63} - \sqrt{252} - \sqrt{700} + \sqrt{5}$
6. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} + \sqrt{\frac{5}{8}}$
7. $\sqrt[3]{27n^2} - \sqrt[3]{125n^2} + \sqrt[3]{216n^2}$
8. $\sqrt[3]{a} + \sqrt{4a} - \sqrt{9a} + \sqrt[3]{8a}$

3. MULTIPLICATION OF RADICALS.

124. Multiply $\sqrt{2}$ by $\sqrt{3}$ and give the result correct to two decimal places.

Can you find a better way of doing it?

(If you cannot hit upon a better way, it may be that you will get hold of an idea if you express the radicals in the exponential form and then use a certain one of the exponential laws.)

Now formulate a rule for the multiplication of radicals.

Apply to $\sqrt{5} \times \sqrt{15}$, test your result. Which method do you prefer? Why?

Perform the operations indicated and simplify the results.

1. $\sqrt{24} \cdot \sqrt{6}$
2. $\sqrt{3} \cdot \sqrt{2} \cdot \sqrt{30}$
3. $\sqrt{3a} \cdot \sqrt{5a}$
4. $\sqrt{2n^3} \cdot \sqrt{5n}$

5. $\sqrt{2ab^3}\cdot\sqrt{2ab^3}$

6. $3\sqrt{2a}\cdot 4\sqrt{3ac}$

7. $\sqrt[3]{7a^2}\cdot\sqrt[3]{a^5}$

8. $\sqrt{x+y}\cdot\sqrt{x-y}$

9. $\sqrt{a+b}\cdot\sqrt{(a+b)^3}$

10. $\sqrt{x-y}\cdot\sqrt{x^2+xy+y^2}$

11. $\sqrt{\dfrac{x+y}{x-y}}\cdot\sqrt{\dfrac{x+y}{x-y}}$

12. $\sqrt{\dfrac{a^2-b^2}{a^2+b^2}}\cdot\sqrt{\dfrac{a^2+b^2}{a-b}}$

13. $(\sqrt{2}+\sqrt{3})\,(\sqrt{2}-\sqrt{3})$

14. $(\sqrt{2}+\sqrt{3})^2$

15. $(3\sqrt{3}+2\sqrt{2})\,(4\sqrt{3}-3\sqrt{2})$

16. $(\sqrt{a}+\sqrt{b})\,(2\sqrt{a}-3\sqrt{b})$

17. $(2\sqrt{6}+\sqrt{3}-7\sqrt{2})\,(\sqrt{6}-2\sqrt{3}+4\sqrt{2})$

18. $(-1+\sqrt{3})^2$

19. $(\sqrt{a+b}+a)\,(\sqrt{a+b}-a)$

20. $\sqrt{n+\sqrt{r}}\cdot\sqrt{n-\sqrt{r}}$

21. $(\sqrt{2a^3b})^2$, $(4\sqrt{2ab^3c^5d})^3$, $(2\sqrt{a^3cy})^5$

22. $(\sqrt{2}+\sqrt{3})^2\cdot(\sqrt{2}+\sqrt{3})^3$

23. $(n+\sqrt{r})^2-(n-\sqrt{r})^2$

24. $\sqrt{2}\cdot\sqrt[3]{2}$. In what does this exercise differ from those given above? Can you find a way to perform the multiplication. (Suggestion:— Again make use of the exponential form and the laws of exponents.)

25. $\sqrt{a}\cdot\sqrt[3]{a}$

Such expressions do not often occur. It is better to use the exponential form in such cases.

4. DIVISION OF RADICALS.

125. Show that $\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{a\div b}$

Formulate rule and apply to the following.

1. $\sqrt{3xy^3}\div\sqrt{xy}$

2. $\sqrt{6abcd}\div\sqrt{2cd}$

3. $\dfrac{\sqrt{2}}{\sqrt{3}}$, $\dfrac{\sqrt{a}}{\sqrt{b}}$, $\dfrac{\sqrt{7}}{\sqrt{8}}$, $\dfrac{\sqrt{2n}}{\sqrt{3n^2}}$

In most cases of division it is better to proceed as follows:—

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}\,.\,\sqrt{3}}{\sqrt{3}\,.\,\sqrt{3}} = \frac{\sqrt{6}}{3}$$

so

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\,.\,\sqrt{b}}{\sqrt{b}\,.\,\sqrt{b}} = \frac{\sqrt{ab}}{b}$$

and

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b^2}} = \frac{\sqrt[3]{a}\,\sqrt[3]{b}}{\sqrt[3]{b^2}\,\sqrt[3]{b}} = \frac{\sqrt[3]{ab}}{b}\,.$$

We have rationalized the denominator by multiplying the terms by a properly chosen multiplyer.

Perform the divisions.

1. $2\sqrt{5} \div \sqrt{6}$,
2. $2\sqrt{6} \div 7\sqrt{12}$
3. $\sqrt{6a^3bc} \div 3\sqrt{4a^3b^2c}$
4. $(2\sqrt{6} - 10\sqrt{2} + 6\sqrt{18} + 3) \div \sqrt{3}$
5. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$, $\frac{2}{\sqrt{3}}$, $\frac{5}{\sqrt{5}}$, $\frac{a}{\sqrt{a}}$, $\frac{5a}{3\sqrt{2a}}$
6. $\frac{1}{\sqrt[3]{2}}$, $\frac{2}{\sqrt[3]{3}}$, $\frac{\sqrt{2}}{\sqrt[3]{3}}$, $\frac{\sqrt{3}}{\sqrt[3]{3}}$, $\frac{3\sqrt{a}}{\sqrt[3]{a^2}}$

12. $\frac{2}{\sqrt{8}}$, $\frac{3ab}{3\sqrt{18}}$, $\frac{\sqrt{3}}{3\sqrt{27}}$, $\frac{1}{\sqrt{a^5}}$

8. $(\sqrt{8a} + \sqrt{12a} + \sqrt{2a}) \div \sqrt{2a}$

126. When the divisor is a polynominal containing radicals the same method of simplification must be followed.

Can you apply the method to $\frac{1}{\sqrt{a} + \sqrt{b}}$?

Find the value of $\frac{1}{\sqrt{3} + \sqrt{2}}$ by two methods giving the result correct to two decimal places.

Will both the methods used in the last exercise apply to the more general case,

$$\frac{1}{\sqrt{a} - \sqrt{b}}$$

State a rule for the rationalization of the denominator of a fraction.

$\sqrt{a} + \sqrt{b}$ is called the conjugate of $\sqrt{a} - \sqrt{b}$.

Reduce the following to equivalent fractions with rational denominators. In all cases where definite numbers are given find the numerical value to two decimal places.

1. $\dfrac{1}{\sqrt{3} - \sqrt{2}}$

2. $\dfrac{1}{\sqrt{a} - \sqrt{b}}$

3. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

4. $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$

5. $\dfrac{12}{\sqrt{5} + \sqrt{2}}$

6. $\dfrac{1}{2\sqrt{3} - 3\sqrt{2}}$

7. $\dfrac{2}{2 - \sqrt{2}}$

8. $\dfrac{\sqrt{3}}{\sqrt{3} - 2}$

9. $\dfrac{x - y}{\sqrt{x} - \sqrt{y}}$

10. $\dfrac{9 - 5\sqrt{3}}{7 - 3\sqrt{3}}$

11. $\dfrac{2\sqrt{5} + 3\sqrt{2}}{2\sqrt{5} - 4\sqrt{2}}$

12. $\dfrac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}}$

13. $\dfrac{a + \sqrt{a^2 - 3}}{a - \sqrt{a^2 - 3}}$

14. $\dfrac{x^2}{y + \sqrt{y^2 - x^2}}$

15. $\dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

16. $\dfrac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$

17. $\sqrt{\dfrac{2 + \sqrt{2}}{2 - \sqrt{2}}}$

18. $\sqrt{\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}}}$

19. Simplify $\left(\sqrt{\dfrac{a+1}{a}} - \sqrt{\dfrac{a}{a+1}}\right)^2$

20. Express in one term $\dfrac{a}{x}\sqrt{\dfrac{a^2}{x^2} + 1}$

21. Simplify

$$\left(\frac{1 - \sqrt{3}}{2} + 1\right)\left(\frac{1 - \sqrt{3}}{2} - 1\right)\left(\frac{1 - \sqrt{3}}{2} - 2\right)^2$$

127. With the help of irrationals we can factor many expressions which we could not factor before.

$$x^2 - 2 = (x - \sqrt{2})\ (x + \sqrt{2})$$

Factor

1. $9x^2 - 8$
2. $3a^2 - 5$
3. $3t^2 - 49$
4. $r^2 - 7$
5. $x^2 - 50$
6. $x^2 + 7$

128. $(x + 2 + \sqrt{3})\ (x + 2 - \sqrt{3}) = ?$

Notice that

$$\begin{aligned}(x + 2 + \sqrt{3})\ (x + 2 - \sqrt{3}) &= (x + 2)^2 - (\sqrt{3})^2 \\ &= x^2 + 4x + 4 - 3 \\ &= x^2 + 4x + 1\end{aligned}$$

Reversing this work we can factor $x^2 + 4x + 1$

$$\begin{aligned}x^2 + 4x + 4 + 1 - 4 &= x^2 + 4x + 4 - 3 \\ &= (x + 2)^2 - 3 \\ &= (x + 2 + \sqrt{3})\ (x + 2 - \sqrt{3})\end{aligned}$$

4 has been added so that with it the first two terms shall be a trinomial square. 4 must then be subtracted to restore the original value. This is called completing the square.

Factor

1. $x^2 + 6x + 4$
2. $x^2 - 2x - 5$
3. $x^2 - 8x - 3$
4. $x^2 + 3x - 3$
5. $4x^2 - 6x + 1$
6. $2x^2 + 10x - 3$
7. $3x^2 - 4x + 1$
8. $x^2 + 2x + 5$

129. As has been said the laws of exponents are applicable to expressions having exponents of any sort whatsoever. We have had some experience with fractional exponents a little way back. You will often come upon such expressions as 2.37×10^{-5} in which there is a negative exponent. What does it mean?

You remember our law of exponents for division.

$$a^5 \div a^3 = a^{5-3} = a^2$$

so also

$$a^5 \div a^7 = a^{5-7} = a^{-2}$$

So that a^{-2} is the quotient of a^5 by a^7. But this quotient found in the other way (casting out common factors) is $\frac{1}{a^2}$ therefore it follows that $a^{-2} = \frac{1}{a^2}$

And we see that a negative exponent indicates that the number with which it is affected is to be used as a divisor.

2.37×10^{-5}, then, means $2.37 \times \frac{1}{10^5}$ or as you have been more accustomed to write .0000237

Interpret and find numerical values.

1. 3^{-2} , 2^{-3} , 5^{-2} , 10^{-1} , 1^{-1}
2. $4^{-\frac{1}{2}}$, $8^{-\frac{1}{3}}$, $16^{-\frac{1}{4}}$, $4^{-.5}$

The following table is of interest

$$a^3 = 1.a.a \cdot a$$
$$a^2 = 1.a.a$$
$$a^1 = 1.a$$
$$a^0 = 1.$$
$$a^{-1} = 1 \div a$$
$$a^{-2} = 1 \div a \div a$$
$$a^{-3} = 1 \div a \div a \div a$$

Interpret

1. ab^{-1} , $a^{-1}b$, $a^{-1}b^{-1}$, $a^{-1}b^6$
2. 2.7×10^{-7} , 14.71×10^{-10}
3. a^2b^{-3} , $x^{-5}y^7$, $x^0y^3z^{-0}$

130. As we have seen, a fractional exponent denotes an irrational; a negative root denotes a fraction. If this is kept in mind there is little difficulty in interpreting results of operations involving such exponents.

A few illustrations

$$2x^3 . 2x^{-2} = 6x^{3-2} = 6x$$

$$(2a^3x^{-3})^{-2} = 2^{-2}a^{-4}x^6 = \frac{x^6}{4a^4}$$

$$a^{-\frac{1}{3}}.a^{\frac{4}{3}} = a^{-\frac{1}{3}+\frac{4}{3}} = a$$

$$\frac{a^{-1}b^3}{xy^{-2}} = \frac{b^3y^2}{xa}$$

$$\frac{x^{-1}+y}{y} = \frac{\frac{1}{x}+y}{y} = \frac{1+xy}{xy}$$

Perform the operations indicated and interpret results.

1. $2^{-3}a \times 2a^5$
2. $x^0y \times x^2y^{-3}$
3. $n^{-2}n^3n^{-4}$
4. $3^{-2}b^0.b^2.9b^{-1}$
5. $2a^3x.4a^{-1}x$
6. $3ax^{-2}.2a^{-3}x$
7. $4x^{-3}.5ax^2y^0$
8. $x^{-0}+y^0$
9. $r^{1-n}.r^{1+n}$
10. $t^{n-4}t^{2-n}t^{n-2}t^4$
11. $2x^2y^{-3}.4x^{-1}y^3z$
12. $4x^{-2}y^3z.4x^2y^{-3}z^{-1}$
13. $(x^{-2}+x^2)^2$
14. $(x^{-2}+y^{-2}) = (xy)$
15. $(x^{-2}+y^{-2}) = (x^{-2}-y^{-2})$
16. $(a^{-1}+b^{-1})\ (a-b)$
17. $(a^{-1}+b)^0$
18. $(a^{-1}+b)^{-1}$
19. $9a^{-2}b^0c^7/3a^2b^{-3}c^6$
20. $27a^{-5}b^{-6}c^3/5a^{-5}b^{-9}c^{-1}$
21. $(x-y)^{-2}/(x-y)^{-3}$
22. $28r^{-3}s^{-3}t/7r^2s^{-7}t^{-4}$
23. $(3a^{-1}b^3)^{-2}$
24. $(5x^2y^{-3})^2$
25. $((2ab^2c^{-1}d^{-3})^{-2})^{-1}$
26. $(5x^{-5}y^6z^3w^{-4})^{-3}$
27. $((s^{-2}h^{-3}k^4)^3)^{-2}$
28. $(a+b)^{-2}$
29. $(a^{-2}+b^{-2})^{-2}$
30. 2.31×10^{-5}
31. $(x^{-2}+x^{-1}+2+2x+x^2)\ (x^{-1}+2)$
32. $3^{-2} \times 27 \times 12^{-1} \div 2^2$
33. Factor $x^{-2}-y^{-2}$
34. Factor $a^{-2}+2+a^2$
35. $5a^2b^{\frac{1}{2}} \times 3a^{\frac{1}{2}}b^{\frac{3}{2}}$
36. $3n^{-\frac{3}{4}}/5n^{-\frac{5}{6}}$
37. $(a^{\frac{1}{2}}+b^{-\frac{1}{2}})^2$
38. $e^{\frac{t}{2}}(ate^{-\frac{t}{2}}+2ae^{-\frac{t}{2}}+c)$
39. $(3x^{2n-1}9^{-1}x^{-n}) \div (2x^{-1}.5x^{1-n})$

Reduce to forms free of negatives exponents.

40. $\dfrac{3x^{-1}y}{5ab^{-2}}$

41. $\dfrac{a^{-\frac{1}{2}}b^{3}}{a^{-\frac{3}{2}}b^{-2}}$

42. $\dfrac{a^{-1}+b^{-1}}{ab}$

43. $\dfrac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$

44. $\dfrac{a^{-2}-b^{-3}}{a^{-1}+b^{-1}}$

45. $\dfrac{a^{-3}-b^{-3}}{a-b}$

46. $\dfrac{x^{-3}-b^{-3}}{x^{3}-b^{3}}$

47. $\dfrac{1-x^{-2}}{1-x^{2}}$

48. $(a^{-1}+b^{-1})^{3}/(a+b)^{3}$

49. $e^{-x}+e^{x}$

50. $\dfrac{e^{-x}-e^{x}}{e^{-x}+e^{x}}$

51. $\dfrac{1+e^{x}}{e^{-x}+1}$

52. Show that $\dfrac{e^{-\frac{x}{2}}-e^{\frac{x}{2}}}{2})^{2}+1=(\dfrac{e^{-\frac{x}{2}}+e^{\frac{x}{2}})^{2}}{2}$

53. Express without negative exponents, $\dfrac{x^{\frac{1}{3}}-a^{\frac{2}{3}}x^{-\frac{1}{3}}}{a^{\frac{2}{3}}x^{-\frac{1}{3}}}$

54. Free of negative exponents $\dfrac{x^{\frac{1}{3}}+a^{\frac{2}{3}}x^{-\frac{2}{3}}}{1+a^{\frac{2}{3}}x^{-\frac{2}{3}}}$

55. Free of negative exponents by multiplying both terms of the fraction by $(x-2)^{\frac{1}{2}}$

$$\frac{(x-2)^{\frac{1}{2}}+2(x-2)^{-\frac{1}{2}}}{(x-2)^{\frac{1}{2}}}$$

56. $((b^{n+m})^{n})(^{n+m})$

57. $(a^{2}-b^{2})^{a-b}\div(a+b)^{a-b}$

58. $\dfrac{a^{x-1}b^{-x-2}c^{x-3}}{x^{n-1}y^{n-2}z^{n-3}}\cdot\dfrac{x^{n+1}y^{n-3}z^{n+4}}{a^{x-2}b^{x+1}c^{x-1}}\cdot\dfrac{a^{2}c^{2}}{b^{2}x^{2}}$

59. $\dfrac{a^{n}+a^{r}}{a^{n}-a^{r}}+\dfrac{a^{n}-a^{r}}{a^{n}+a^{r}}-\dfrac{a^{2n}+(a^{2})^{r}}{(a^{n})^{2}+a^{2r}}$

60. $\dfrac{a^{2x-3y}}{a^{3x-2y}}\cdot\dfrac{a^{3x-y}}{a^{x-3y}}$

61. $\dfrac{2a^{n}b^{m-1}c^{3}}{4a^{m-n}b^{n-m}c^{0}}\div\dfrac{6a^{2-m}}{b^{n}c^{-n}}$

CHAPTER 9.

LOGARITHMS

131. But we have not yet reached the end of the uses to which the exponent may be put. We are about to begin an investigation which will dispel much of the dreariness of numerical calculations as with snap of the finger.

132. Consider the following table of the powers of 2.

$2^0 = 1$	$2^5 = 32$	$2^{10} = 1024$	$2^{15} = 32768$
$2^1 = 2$	$2^6 = 64$	$2^{11} = 2048$	$2^{16} = 65536$
$2^2 = 4$	$2^7 = 128$	$2^{12} = 4096$	$2^{17} = 131072$
$2^3 = 8$	$2^8 = 256$	$2^{13} = 8192$	$2^{18} = 262144$
$2^4 = 16$	$2^9 = 512$	$2^{14} = 16384$	$2^{19} = 524288$

We know that $2^5 . 2^7 = 2^{5+7} = 2^{12}$ or if we use the equal numbers from the table.

$$32 \times 128 = 4096$$

Verify this by long multiplication

$$256 \times 128 = ?$$

This can be done in two ways by mutiplication or thus.

$$256 = 2^8$$

$$128 = 2^7$$

$$256 \times 128 = 2^8 . 2^7 = 2^{8+7} = 2^{15} = 32768$$

for from the table it is seen that

$$2^{15} = 32768$$

Thus the use of the table saves a lot of computation.

1. $64 \times 256 = ?$
2. $4096 \div 64 = ?$
3. $32768 \div 512 = ?$

4. $4096 \times 16384 \div 262144 = ?$

5. $(128)^2 = ?$

It will be noticed that multiplication and division have been replaced in each case by addition and subtraction of the exponents.

But the method appears to be restricted to numbers which are powers of 2. If it were possible to express all numbers in terms of 2 with some exponent the method would be of more general use. You can see at once that the table can be extended by adding such numbers as

$$2^{\frac{1}{2}} = 1.414+ \quad , \quad 2^{\frac{3}{2}} = 2.828+$$

$$2^{-1} = .5 \qquad 2^{-2} = .25, \text{ etc.}$$

And it is at once evident that such extension is unlimited, and that a very useful table could be so constructed.

133. We could use any positive numbers other than 2 as the base of such a table. But since the base of our system of Arithmetic is 10, it has been found most convenient to use 10 as a base of such a table, thus

$$\begin{aligned} 10^0 &= 1 \\ 10^1 &= 10 \\ 10^2 &= 100 \\ 10^3 &= 1000 \quad \text{etc.} \end{aligned}$$

By the use of fractional exponents we may fill in the table thus.

$$\begin{aligned} 10^0 &= 1 \\ 10^{.5} &= 3.162 \\ 10^1 &= 10 \\ 10^{1.5} &= 31.62 \\ 10^2 &= 100 \quad \text{etc.} \end{aligned}$$

Since the base used is always 10 we may omit it and write the table in two columns, a column of exponents and a column of numbers to which the exponent belongs.

EXPONENTS.	NUMBERS.
0	1
.5	3.162
1.	10
1.5	31.62
2.	100.
2.5	316.2
3	1000.

We could fill in even closer.

$$10^{.25} = 10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = (3.162+)^{\frac{1}{2}} = 1.77+$$

and so on.

You will note the regularity of the table. The exponent of any number between 1 and 10 is between 0 and 1, the exponent of any number between 10 and 100 is between 1 and 2 and so on.

It is also to be noted in the table we have constructed that the exponents are expressed in exact numbers while most of the numbers are given only approximately. Any one would rightly judge that this is not very convenient for use. The tables for actual use then, are made with the numbers exact and in regular order while the exponents belonging to them are given approximately.

But the method of computing such a table is different from that we have just used. That, you must leave until some later date when you have acquired more knowledge of algebra.

134. It is the common custom to call these exponents by another name—logarithm; thus

$10^3 = 1000$ 3 is the logarithm of 1000. This is stated in the more elegant form $\log 1000 = 3$.

But always remember that logarithm is but another name for an exponent.

What is the logarithm of 10, of 100, of 100000 of 1, of .001?

Computers have found that

$$10^{.4771} = 3$$

so that $\log 3 = .4771$

Below is given a little of the table of logarithms that has been carefully computed.

NUMBER.	LOGARITHM.	NUMBER.	LOGARITHM.	NUMBER.	LOGARITHM.
1	0.	10	1.	100	2.
2	.30103	20	1.30103	200	2.30103
3	.47712	30	1.47712	300	2.47712
4	.60206	40	1.60206	400	2.60206
5	.69897	50	1.69897	500	2.69897
6	.77815	60	1.77815	534	2.72751
7	.84510	70	1.84510		
8	.90309				
9	.95424				

135. Note that since

$$6 = 2 \cdot 3 = 10^{.30103} \times 10^{.47712} = 10^{.30103 + .47712} = 10^{.77815}$$

the logarithms of some numbers can be computed from those of other numbers and thus the table be extended,

What is the logarithm of 12? log 15? log 24?

136. The logarithm of any number between 100 and 1000 will be between 2 and 3. The logarithm of 534 lies between 2 and 3; by computation it has been found to be 2.7275 nearly; the exact value cannot be found.

$$\log 534 = 2.7275$$

What is $\log 53.4$?

$$53.4 = 534 \div 10 = 10^{2.7275} \div 10^{1} =$$
$$10^{2.7275-1} = 10^{1.7275}$$

therefore $\log 53.4 = 1.7275$

so also $\log 5.34 = \log 534 - \log 100$

$$= 2.7275 - 2 = .7275$$

$$\log .534 = 2.7275 - 3$$

$$\log .0534 = 2.7275 - 4$$

$$\log 5340 = 2.7275 + 1 = 3.7275$$

$$\log 53400 = 2.7275 + 2 = 4.7275$$

From this it appears that a logarithm consists of two parts, quite independent of each other, an integral part and a decimal part.

The integral part, to which the name characteristic is given is dependent upon the position of the decimal point.

The decimal part, called the mantissa, is independent of the position of the decimal point and remains the same for a given sequence of the digits.

The mantissas of logarithms can be found in the tables.

The characteristics can be found upon inspection by means of the following rule.

Rule: If the number is greater than unity, the characteristic is positive and is one less than the number of digits to the left of the decimal point.

If the number is less than unity, the characteristic is negative and is equal to the number of the decimal place occupied by the first significant figure.

$$\log 534000 = 5.7275$$
$$\log .0534 = .7275 - 2$$

The mantissa is always positive and when the logarithm has a negative characteristic it is often written, $\bar{2}.7275$, the negative sign being placed above the negative part, $\bar{2}.7275$ means $-2 + .7275$.

What will be the characteristics of the logarithms of 3, 94, 396, 4356, 56785, 456789?

What will be the characteristics of the logarithms of .3, .34, .0876, .005, .000054?

137. In the table on pages 161-163 the mantissas of the logarithms of all integral numbers between 100 and 1000 have been arranged in a form convenient for reference.

In column N are placed the first two figures of the numbers, the third figures are placed at the head of the successive columns.

The mantissa of a number is placed in the row denoted by the first two of its digits and the column denoted by its third digit.

Thus the mantissa of the logarithm of 263 is found in row 26 column 3 and is .4200. The characteristic is 2; hence the log 263 = 2.4200.

138. Since the mantissas depend upon the sequence of figures only, the logarithm of any number between 1 and 100 will be found in column 0.

Find the logarithms of the following numbers.

7, 25, 94, 125,
329, 239, 932, 970,
2.37, 72.3, .253, .0723,
.0023, .00075, .0529, 63200,

139. What is the number whose logarithm is 2.5145? The mantissa is found in row 32, column 7; the characteristic is 2: therefore the number is 327.

Find the numbers corresponding to the following logarithms:

1.3404 , 3.4425 , $.7852$, $\bar{1}.8331$
$\bar{2}.9643$, 1.9079 , 1.0792 , 2.0969
$\bar{3}.2330$, $\bar{1}.1367$, $.5237$, $.5185$

140. Let it be required to find the log 3674. The mantissas in the table are given for numbers of three significant figures. This number has four. We proceed as follows:

The mantissa for log 368 is .5658
The mantissa for log 367 is .564~~8~~7
1 11

A difference of one in the third place of the number corresponds to a difference of 11 in the fourth place of the mantissa. Hence a difference of .4 in the number will correspond to a fference of $.4 \times 11 = 4.4$ in the mantissa, (as the decimal part of this result is .4 is less than .5 we will follow the usual practice and disregard it). We add this correction, 4, to the mantissa of log 367.

Mantissa of log 367 is .5647
4
Mantissa of log 3674 is .5651
Hence log 3674 = 3.5651

The difference between the consecutive mantissas of the table is called the tabular difference.

Find log 52.321.

Mantissa of log 524 = .7193
Mantissa of log 523 = .7185
tabular difference = 8
Correction is 8 × .21 = 1.68
mantissa of log 523 = .7185
.2
mantissa of log 52321 .7187
hence log 52.321 = 1.7187

State a rule for finding the logarithm of a number having more than three significant figures.

Find logarithms of

1235 , 26.75 , 392.4
.4319 , 24031 , .005236
921.32 , 11.239

141. To find the number corresponding to a mantissa not found in the table.

This is the reverse problem and we have but to retrace the steps of the last article.

1.2775 is log of what number? The mantissa 2775 is found to be between the mantissa 2765 and 2788 of the table; the number then must lie between the numbers corresponding to these two mantissas, that is between 189 and 190.

Mantissa 2788 corresponds to no. 190
Mantissa 2765 corresponds to no. 189
Tabular difference 23

The correction to be added to number 189 is found by dividing the difference between the mantissa of the required number and the mantissa of 189 by the tabular difference, thus:—

2775
2765
23|10
.4

adding this to 189 we have 1894.

Thus 1.2775 is the logarithm of 18.94

142. Find the number corresponding to the logarithms.

2.4224 , 1.7816 , .2428
1.6380 , 9607 , 2.5882
1.6380 , .9607 , 2.5882

143.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1071	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010

N	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7887
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445

N	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8844	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	999

144. $327 \times 3.8 = ?$

Regarding these numbers as powers of 10 we have but to add their logarithms and find the number corresponding to that result.

Arrange the work thus:

$$\begin{array}{rl} \log 327 = & 2.5145 \\ \log \ \ 38 = & \underline{\ \ .5798} \\ & 3.0943 = \log 1242.5 \end{array}$$

By long multiplication the product is found to be 1242.6.

Results obtained by the use of logarithms are approximate. With a four place table we can get four or five places correct, which is as close as we desire in most cases. Of course a five place table would give closer approximations and a ten place table even better still.

145. Find the square root of 3.

$$\sqrt{3} = 3^{\frac{1}{2}} = (10^{.4771})^{\frac{1}{2}} = 10^{.2385} = 1.732$$

that is, divide the log of 3 by 2 and find the corresponding number.

$$\log 3 = .4771$$

$$\text{divide by 2 and get } .2385 = \log 1.732$$

We have now that simpler method of finding the cube root of a number referred to sometime back.

Find cube root of 2.

$$\log 2 = .3010$$

$$\text{divide by } 3 = .1003 = \log 1.259$$

$$\text{divide by 3 and get } .1003 = \log 1.259$$

146. Since logarithms are exponents the following statements are at once seen to be true, then bring but restatements of the laws of exponents.

1. The logarithm of a product is equal to the sum of the logarithms of the factors.

2. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

3. The logarithm of the power of a number is equal to the index of the power times the logarithm of the number.

4. The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.

Logarithms, then, may be used in simplifying the operations of multiplication division, involution and evolution.

147. Compute the value of

1. 236×29
2. 3.64×79.3
3. $6.75 \times 7.92 \times 8.32$
4. $7.07 \times 9.01 \div 14.96$
5. $\dfrac{94.32}{100.5} \times 52.63$
6. 369.5×4.953
7. $\sqrt{2}$
8. $\sqrt[3]{5}$
9. $\sqrt[5]{11}$
10. 1.748^4
11. Find area of a circle whose radius is 2.37
12. Find the volume of a sphere whose diameter is found to be 13.29 cm.
13. $2/\pi$
14. $\dfrac{8\sqrt{10}}{17}$
15. The dimensions of a box are 7.2 , 8.9 and 10.7 cm. Find its volume.
16. The base and altitude of a right angled triangle are 395 and 276. Find the hypotenuse.
17. $\sqrt[3]{1.072} \times \sqrt{5.21}$
18. $\dfrac{213 \times 7.655}{3145 \times 718}$

CHAPTER 10.

IMAGINARY NUMBERS, COMPLEX NUMBERS.

148. Find the square root of 4 .

Find the square root of -4 .

There seems to be no answer to the second of those questions.

Solve the equation $x^2-1=0$

Solve the equation $x^2+1=0$

In the solution of the second equation we meet with the same difficulty.

A negative number seems to have no square root. And so men thought for a great many years.

We now consider that a negative number has a square root, tho it is very different in kind from the other numbers with which we have become acquainted.

Just as we use $\sqrt{4}$ or $4^{\frac{1}{2}}$ to denote the square root of 4.

So let us use $\sqrt{-4}$ or $(-4)^{\frac{1}{2}}$ to denote the square root of -4.

We have found an exact value for $\sqrt{4}$.

We have found an approximate value for $\sqrt{2}$.

What can we make out of $\sqrt{-4}$?

Putting $\sqrt{-4}$ in the exponential form we have:—

$$\sqrt{-4}=(-4)^{\frac{1}{2}}=(4.-1)^{\frac{1}{2}}$$
$$=4^{\frac{1}{2}}.(-1)^{\frac{1}{2}}=2\sqrt{-1}$$

and so in general

$$\sqrt{-a}=(-a)^{\frac{1}{2}}=(a\cdot-1)^{\frac{1}{2}}=a^{\frac{1}{2}}.(-1)^{\frac{1}{2}}=\sqrt{a}\sqrt{-1}$$

So that the square root of any negative number may be expressed as some number times $\sqrt{-1}$, and we can regard $\sqrt{-1}$ as a sort of unit; just as 1 is regarded as the unit of the natural numbers.

This new kind of number, the square root of a negative number is called imaginary number.

All other kinds are called real numbers.

These imaginary numbers have been known and used for but a few hundred years. The Hindus did not use them at all. In the sixteenth century the Italians called them "impossible numbers." The name "imaginary" is due to Descartes who lived from 1596—1650.

Write in the new form so as to display the unit:—

$$\sqrt{-9}\ ,\ \sqrt{-8}\ ,\ \sqrt{-16}\ ,\ \sqrt{-a^2}\ ,\ \sqrt{-ab^2}$$

149. Imaginary numbers are of very great service in mathematical work.

They have certain peculiarities of their own which must be understood if we are to use them correctly.

By definition $\sqrt{-1}\,.\sqrt{-1}=-1$

A careful study of the table below will make clear the chief peculiarity of imaginary number.

$$\sqrt{-1}=\sqrt{-1}\ ,\ (\sqrt{-1})^5=\sqrt{-1}$$
$$(\sqrt{-1})^2=-1\ ,\ (\sqrt{-1})^6=-1$$
$$(\sqrt{-1})^3=-\sqrt{-1}\ ,\ (\sqrt{-1})^7=-\sqrt{-1}$$
$$(-1)^4=1\ ,\ (\sqrt{-1})^8=1$$

What will be the value of $\sqrt{-1}$ to the $4n$, $4n+1$, $4n+2$, $4n+3$ powers?

It is seen that the powers of the unit of imaginaries have a range of four different values that repeat themselves in regular order. This must be taken into account whenever we make computations with them.

150. In operating with imaginary numbers it is well to separate the units, especially is this so in multiplication.

$$\sqrt{-2}\ \sqrt{-3} = \sqrt{2}\ \sqrt{-1}\ \sqrt{3}\ \sqrt{-1} = \sqrt{2}\ \sqrt{3}\ \sqrt{-1}\ \sqrt{-1}$$
$$= (\sqrt{-1})^2\sqrt{6} = -\sqrt{6}$$

If the peculiarity of the powers of the unit be remembered, imaginaries are combined in the same way as are radicals.

Perform the indicator operations.

1. $\sqrt{-2} \cdot \sqrt{-5}$
2. $\sqrt{-5} \cdot \sqrt{3}$
3. $\sqrt{-2}\ \sqrt{-3}\ \sqrt{-8}$
4. $\sqrt{-18a}\ \sqrt{-50a}$
5. $(\sqrt{-3} + \sqrt{-2})\ \sqrt{-3}$
6. $\sqrt{-5}\ /\ \sqrt{-3}$
7. $(\sqrt{-a} + \sqrt{-b})\ (\sqrt{-a} - \sqrt{-b})$
8. $1\ /\ \sqrt{-25}$
9. $\sqrt{-20} - \sqrt{-5}$
10. Show that $\left(\frac{-1+\sqrt{-3}}{2}\right)^3 = \left(\frac{-1+\sqrt{-3}}{2}\right)^3$
11. Show that $\left(\frac{-1+\sqrt{-3}}{2}\right)^3 = \left(\frac{-1+\sqrt{-3}}{2}\right)^9$
12. $\frac{2+\sqrt{-2}}{4}$
13. $\frac{1-\sqrt{-2}}{1+\sqrt{-2}}$

The unit $\sqrt{-1}$ is often represented by the letter i. Thus:—

$$\sqrt{-2} = \sqrt{2}\ \sqrt{-1} = i\sqrt{2}$$

This is a very convenient notation and quite universal among mathematicians. It is due to a German professor, Gause. (1777—1855.)

14. $i^3 \div i$
15. i/i^3
16. $1/i^5$
17. $(i + i^3)\ (i^2 - i)$
18. $(i\sqrt{2} + i)\ (i\sqrt{2} + 3i)$

The sum of an imaginary and a real number is called a complex number, thus:—

$$2+\sqrt{-5} \text{ or } 2+i\sqrt{5}$$

or in general $a+ib$ where i is the unit of imaginaries.

Express the following in the form of complex numbers.

19. $1+2i-3i-(5-i)+(7-2i)$

20. $(2-3i)\ (3+4i)$

21. $(5-i)\ (8+3i)$

22. $(a+ib)^2-(a-ib)^2$

23. $(a+ib)\ (a-ib)+(a^2+b^2)$

24. $\dfrac{4}{1+i}\ ,\ \dfrac{100}{3+4i}\ ,\ \dfrac{4}{i-1}$

25. $\dfrac{7}{-3+\sqrt{-5}}\ ,\ \dfrac{1-\sqrt{-2}}{1+\sqrt{-2}}$

26. $\dfrac{1+i}{1-i}\ ,\ \dfrac{p+iq}{p-iq}$

27. $\dfrac{3+\sqrt{-5}}{4-\sqrt{-7}}$

28. $\dfrac{(a+ib)^2}{a^2-b^2+i(2ab)}$

29. Factor a^2+4, Use imaginaries.

$$\begin{aligned} a^2+4=a^2-(-4) &= (a+\sqrt{-4})\ (a-\sqrt{-4}) \\ &= (a+2i)(a-2i) \end{aligned}$$

30. Factor x^2+9

31. Factor x^2+2

32. Factor x^2+1

33. Factor $2x^2+3$

34. Factor x^2+2x+5

CHAPTER 11.

EQUATIONS OF ONE VARIABLE.

151. We now return to the study of the equation with the purpose of giving it a more systematic treatment.

An equation asserts that two expressions involving numbers are the same in value. The equation is a very powerful thing and enables us to solve many an important problem. It is of such vast importance that modern civilized life could not get along without it.

Suppose you were told that you could have all the gold you could carry away in an open box made from a piece of tin 6 by 9 inches, by cutting out the corners and folding up the sides. I have no doubt you would want to have the very largest box that could be made out of the tin. If x denote the length of the side of the square cut out of the corner, the answer to the problem can be had by solving the equation $2x^2 - 10x + 9 = 0$.

If you were to ask me how this equation is obtained I will have to say that you must wait several years to understand it when it is given. You have a lot of mathematics to learn yet before you can answer that question.

I suspect that none of you can even solve the equation although you have already solved many quadratic equations.

The matter of the gold sounds a good deal like a fairy tale. But suppose it were a box manufacturer who had a lot of straw boards of that size which he wished to make into the largest possible boxes. Or it might be required to find the most economical speed at which to drive a steamboat against a given current.

Or you might want to know what size to make a beam to sustain a certain weight. Or you might want to find out the width of a certain river. All these questions can be answered by solving certain equations.

The important thing to see here is, that the answer to many a problem you may wish to know hangs upon the solution of an equation. It is, then, of the utmost importance to learn how to solve different sorts of equations. There are many kinds of equations and they are classified in various ways. In the time allowed us we can study only a very few kinds.

152. There are two general classes of equations, examples of which we have already had.

$$a^2 - b^2 = (a - b)\ (a + b)\ .$$
$$c = 2\pi r\ ,\ x^2 - x = 6$$

In the first of these equations the two members can be made exactly alike.

Expand the right hand number and we have

$$a^2 - b^2 = a^2 - b^2$$

the two numbers are the same, they are identical. Equations in which this can be done are called identities. And of course the equation is true whatever numerical values may be assigned to the letters involved.

But the equation $c = 2\pi r$ is quite different. The two members cannot be reduced to the same form. It is true that any value may be given to r, but when that is done the value of c becomes fixed. Thus, there is a condition attached to the values that may be given to the letters. $x^2 - x = 6$ is also a case of an equation whose members cannot be reduced to the same form. It is to be further noted, in this case, that there are only two values that can be given to the letters involved. The last two equations are examples of what are sometimes called equations of condition. Tho usually the name equation is sufficient. It is this class of equations we are to consider in this chapter.

153. We have called such an expression as $x^2 - 3x + 2$ a function of x, for it changes its value as the value of x varies: x is called a variable.

$x^2 - 3x + 2$ is a function of one variable.

$x^2 - 4xy - 3y$ is a function of two variable.

$x^2 - 3x + 2 = 0$ is an equation of one variable.

so also is $3x + 2 = 7x - 4$.

The solution of the equation consists in finding for what value of the variable the equation is true.

If the permissible value or root of the equation be substituted for the variable the resulting equation can be reduced to an identity. The root is thus said to satisfy the equation.

It will be noted that what we have heretofore called the unknown, we now call the variable. The other numbers involved will be called constants.

This chapter will be confined to the study of equations of one variable.

154. The simplest equations are those in which the variable has no fractional or negative exponents. Such equations are called rational integral equations to distinguish them from irrational and fractional equations, which contain irrationals and fractions.

$x^2 + x = 3$ is a rational integral equation.

$x + \sqrt{x} - 5 = 0$ is an irrational equation.

$\frac{1}{x} + \frac{2x}{x-1} + x = 5$ is a fractional equation.

We will consider the first kind.

Such equations are classified according to the degree of the term of highest degree.

$3x + 2 = 5x$ is an equation of the first degree and is called a simple or linear equation.

$x^2+3x=5$ is an equation of the second degree and is called a quadratic equation.

$x^3-3x=4$ is of the third degree and is called a cubic.

We treat simple equations first.

1. Simple Equations.

155. Recalling the work of Chapter 1, solve

1. $3x+13-5(x-3)=4(x+3)-1$

2. $\frac{3n}{5}+7-2n=1/2-n$

Write a rule for solving simple equations.

Find the roots of the following equations.

3. $4x+5=5(x+1)$

4. $81-4(y+1)=y+7$

5. $2(9-x)+5(2x+3)=81$

6. $5+n/2=n-5$

7. $200(t-5)=50(80-t)$

8. $10(3+d)+14d=2(2-d)$

9. $19.5y+60=450$

10. $4(1-.1x)=2-.2x+2(1-.3x)$

11. $3=\frac{4}{3}\left(\frac{1}{3}t-\frac{1}{4}\left(\frac{1}{2}t+\frac{1}{6}t\right)\right)$

12. $ax+b=bx+a$ Solve for x

13. $a(x-a)+b(x-b)+c(x+a)=2ab$

14 $\frac{x+b}{b}-\frac{2x}{a}=\frac{a-bx}{ab}=\frac{1+a}{a}$

15. $a(x-b)=b(x-a)$

16. $wa+pc-pb=0$ Solve for p

17. $S=\frac{1}{2}w\left(1-\frac{x}{a}\right)$, Solve for x

156. It will be noticed that in solving these equations we have changed the equations by transposing and by multiplying and dividing both members by the same number. But in no case have we in any way altered the roots of the equation. It will be found that the roots of each succeeding equation will be the same as those of the equation started with so long as the multiplier or divisor do not contain the variable.

The effect of multiplying or dividing by a number containing the variable will be noticed later.

Equations that have precisely the same roots are often called equivalent equations.

Any simple equation can be solved.

2. QUADRATIC EQUATIONS.

157. Any quadratic equation can be put into the standard form

$$ax^2 + bx + c = 0$$

where the terms of the same degree have been collected and the coefficients a, b, c, are fixed numbers.

In Chapter 2 we have seen how to solve certain cases of quadratics.

It was found that many equations could be easily solved by factoring.

$$x^2 + 3x - 28 = 0$$
$$(x - 4)\ (x + 7) = 0$$
$$x = 4\ ,\ x = -7$$

In cases where the term of the first degree was missing it was found to be simpler to transpose the constant term to one de and take the square root of both members.

$$2x^2 = 9$$

$$x = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{2}\cdot\sqrt{2} = \pm 2.121^+$$

State a rule for solving quadratic equations.

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 9x = 10$
3. $y^2 = 9y$
4. $2n^2 - 15 = 7n$
5. $9t^2 = 42t - 49$
6. $3x^2 = 108$
7. $5n\,(n + 9) = 950$
8. $t\,(t + 7) = 7\,(t + 28)$
9. $(x - 2)\ (x + 2) = 12$
10. $x^2 + ax + bx + ab = 0$
11. $(r + 4)\ (r + 5) = 2\ (r + 2)\ (r + 4)$

Will your rule apply to all quadratic equations? Apply to the following:—

12. $x^2 - 4x + 2 = 0$
13. $x^2 - 2x + 2 = 0$

158. Any quadratic equation can be solved by factoring. But oftentimes it will be found to be more convenient to proceed as follows:—

$x^2 - 4x + 2 = 0$

Transpose $x^2 - 4x = -2$

Complete the square of the left member by adding 4 $x^2 - 4x + 4 = 2$

Take square root $x - 2 = \pm\sqrt{2}$

Transpose $x = 2 \pm \sqrt{2}$

$= 3.41$ or $.59$

This amounts to the same thing as factoring but is simpler to apply.

If the quadratic term has a coefficient other than unity divide through by that coefficient.

Find the roots of the following equations by completing the square.

1. $x^2 + 4x = 32$
2. $n^2 - 2n - 1 = 0$
3. $x^2 + 4x = 1$
4. $2x^2 - 9x + 9 = 0$
5. $9a^2 - 9a + 2 = 0$
6. $6y^2 - 37y = -6$
7. $2t^2 = 7t + 11$
8. $x^2 + x + 1 = 0$
9. $t^2 + 32 - 10t = 0$
10. $36x^2 + 3 = 36x$
11. $(x + 2)^2 = 4\,(x - 1)^2$
12. $9x^2 - 6x - 80 = 0$
13. $x^2 - 2x = a^2 + 1$
14. $ax^2 + (a^2 + 1)\,x = -a$

15. Show that the total surface of a circular cylinder is given by the formula

$$S = 2\pi r^2 + 2\pi rh$$

when S = surface, r = radius of base and h equals altitude.

The surface of a cylindrical tin box is to be 400 square cm.; the altitude is to be 8 cm. What must the radius be?

16. $x^2 - 2.3x = 4.5$

17. $16.1t^2 + 10t = 50$

18. $16.1t^2 - 49t = 100$

19. $ax^2 + bx + c = 0$

159. The last equation, the 19th, is the most general form of the quadratic,

$$ax^2 + bx + c = 0$$

Solve by completing the square

Divide thru by a $\quad x^2 + \frac{a}{b}\,x + \frac{c}{a} = C$

transpose $\quad x^2 + \frac{b}{a}\,x = -\frac{c}{a}$

complete square $\quad x^2 + \frac{b}{a}\,x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

$$= \frac{b^2 - 4ac}{4a^2}$$

take square root $$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

transpose and add $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The roots are given in terms of the coefficients of the equation. This constitutes the solution of the most general quadratic.

The result may be regarded as a formula for substitution.

Solve $3x^2-2x-5=0$

$$x=\frac{2\pm\sqrt{2^2+4.3.5}}{2.3}=\frac{2\pm8}{6}=\frac{5}{3}\text{ or }-1$$

Solve by substitution in the formula.

1. $3n^2-10n-8=0$
2. $6n^2-19n+10=0$
3. $x^2-2x=2$
4. $2x^2+5x-1=0$
5. $p^2+p+1=0$
6. $5h^2+3=9h$

160. The general solution shows that all quadratic equations can be solved and that every quadratic equation has two roots.

In fact a quadratic equation has two, and only two, roots. Can you give any reason for the truth of this last statement? If you can see no good reason for it, it is probably better for you to wait for a discussion of the matter in a later course in algebra. It will do you no harm however, to think over it when you have a little leisure.

It is important for you to be able to solve any quadratic that comes to hand.

161. Solve the following equations by the method best adapted to the equation in hand.

1. $9x^2-9x+2=0$
2. $2x^2+9=9x$
3. $3x^2-1=0$
4. $y^2+3y=0$

5. $2n^2 = 1 - 4n$
6. $24t = t^2$
7. $1 + x^2 - x = 0$
8. $n^2 + 7n + 3 = 0$
9. $(4x - 1)\,x = 8$
10. $6t^2 + 2t = 5$
11. $a^2 + 22\,(a + 5) = 0$
12. $n^2 + 90 = 19n$
13. $.07s^2 + .23s = 3.7$
14. $21 + R = 2R^2$
15. $3.2x^2 + 5.2x = 3.7$
16. $\pi r^2 + 6\pi r = 32$
17. $72 = \frac{n}{2}\,(48 + (n - 1)\,(-4)\,)$
18. $350 = 250t - 4.9t^2$
19. $72\,(7 - c^2) + 16c^2 = 0$
20. $(4 + \frac{K}{2})^2 - (16 + 2K) = 0$
21. $16(b - 1)^2 - 80\,(b^2 + 6b + 8) = 0$
22. $21\,(1 + 6c - 3c^2) + 9\,(c + 1)^2 - 0$
23. $x^2 - (1 + \sqrt{3})x + \sqrt{3} = 0$
24. $924 = \frac{22}{21}(r^2 + (9 - r)^2 + r(9 - r)\,)$
25. $11\,(11 + s)\,(11 - s) = 792$
26. $5\,(n - 1)\,(n + 1) = 3\,(3 - n)\,(n + 3)$
27. $x\,(x - 1)\,/4 = x\,(x + 1)\,/12$
28. $(a - 1/3)^2 = 2\,(a - 1/3)$
29. $(a - 1/3)^2 = 1/4$
30. $abx^2 - (a^2 + b^2)x + ab = 0$ Solve for x
31. $x^2 - (a - b + c)x = (a - b)c$ Solve for x
32. $-c^2m^2 + (cm - 2p)^2 = 0$ Solve for c
33. $h^2 + \left(\frac{s}{3}\right)^2 = s^2$ Solve for s
34. $s^2 = \frac{9}{4}\,r^2 + \frac{s^2}{4}$ Solve for s
35. $(r + h)^2 = r^2 + b^2$ Solve for h
36. $q^2x^2 - 2pqx + p^2 - q^2 = 0$ Solve for x

37. $a(x^2+1) = x(a^2+1)$ Solve for x

38. $2\pi rh = 2\pi\left(r^2\frac{1}{4} - h^2\right)$ Solve for h

39. Solve the equation given at the beginning of this chapter in the largest possible box problem.

162. As has been shown the roots of the most general quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The character of these roots will depend upon the character of the radicand $b^2 - 4ac$

If $b^2 - 4ac < 0$, the roots are imaginary
If $b^2 - 4ac = 0$, the roots are real and equal
If $b^2 - 4ac > 0$, the roots are real and unequal
If $b^2 - 4ac$ is a square the roots are rational.

Because of this property of discriminating between the kinds of roots the expression $b^2 - 4ac$ is called the Discriminant of the quadratic.

The discriminant of the equation

$$3x^2 - 7x + 5 = 0$$

is $\quad 7^2 - 4.3.5 = 49 - 60 = -11$

hence the roots are imaginary.

By the use of the discriminant, without solving, determine the character of the roots of the following equations.

1. $9x^2 + 25 + 30x = 0$
2. $x^2 - x + 1 = 0$
3. $x^2 - 2x - 3 = 0$
4. $3x^2 - 3x + 2 = 0$
5. $7x + 5x^2 - 1 = 0$
6. $3x^2 + 7 = 0$
7. $49x^2 - 70x = -25$
8. $2x^2 + 9x + 9 = 0$

9. $9x^2 - 54x + 81 = 0$
11. $5x^2 - 8x + 3 = 0$
10. $5x^2 + 10x + 6 = 0$
12. $(x + 2)^2 = 4x + 15$

13. What is the condition that a quadraitc equation shall have equal roots?

14. In the equation $x^2 + 4x + (1 + k) = 0$ determine k so that the roots shall be equal.

15. In the equation $x^2 + (k + 1)x + \frac{1}{4} = 0$ determine k so that the roots shall be equal.

16. For what values of k are the roots of

$$kx^2 + 2kx - 4x = 2 - 3k$$

real and equal.

17. Show that roots of $ax^2 - (a + b)x + b = 0$ are rational.

163. It is interesting to note that in a quadratic equation in which the coefficient is unity, such as,

$$x^2 + px + q = 0 \qquad (1)$$

that the other coefficients may be expressed in terms of the roots of the equation.

Calling x_1 and x_2 the roots of the general equation

$$ax^2 + bx + c = 0 \qquad (2)$$

we have

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and hence we have

$$x_1 + x_2 = -\frac{b}{a} \text{ and } x_1x_2 = \frac{c}{a}$$

But equation (2) may be reduced to the form of (1) by dividing through by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

so that

$$x_1 + x_2 = -p \quad \text{and} \quad x_1x_2 = q$$

This establishes the theorem:—

When the coefficient of the quadratic term is unity, the coefficient of the linear term equals the sum of the roots with their signs changed, while the absolute term equals the product of the roots.

1. What is the sum of the roots of $3x^2 - 5x = 4$?, the product of the roots?

2. Write an equation such that the sum of its roots shall be 5 and the product shall be 7.

3. Write an equation such that the sum of its roots shall equal the products of its roots.

4. Write an equation whose roots shall be 3 and 4, ; another whose roots are $5, -2$; others with roots, -7, 3 ; $a + b$, $a - b$; $5 + \sqrt{3}$, $5 - \sqrt{3}$.

5. What is the sum of the roots of the three equations?

$$x^2 - 5x + 3 = 0$$
$$x^2 + 5x - 5 = 0$$
$$2x^2 - 8x + 7 = 0 \quad ?$$

3. EQUATIONS OF HIGHER DEGREE.

164. Some few equations of higher degree than the second can be solved by the methods we have considered, but for the most part equations of higher degree than the second must be left until other methods are developed in the higher branches of algebra. We consider two cases.

Some equations can be factored into simple or quadratic factors. Each simple or linear factor gives one root; each quadratic factor gives two roots.

$$x^3 - 1 = 0$$
$$(x - 1)\ (x^2 + x + 1) = 0$$
$$x = 1\ ,\ x = \frac{-1 \pm \sqrt{-3}}{2}$$

Solve

1. $x^3 + 1 = 0$
2. $x^4 - 1 = 0$
3. $x^6 - 1 = 0$
4. $x^3 = 8$
5. $x^3 + 2x = 3x^2$
6. $t^3 + 2t^2 = t + 2$
7. $y^3 - 4y = 4 - y^2$

8 $\frac{4}{3}\pi r^3 = 10$

9. $n^3 - 2n^2 - 5n + 6 = 0$
10. $(n - 1)^3 - 2(n - 1)^2 = 5 - 5n$

165. Some equations of higher degree can be put in the quadratic form

$$ax^{2n} + b^n + c = 0$$

Often such equations can be solved by the quadratic method.

The equation $3x^4 - 2x^2 - 1 = 0$

may be written $3(x^2)^2 - 2(x^2) - 1 = 0$

hence $(3x^2 + 1)(x^2 - 1) = 0$

which gives $3x^2 + 1 = 0$ and $x^2 - 1 = 0$

hence $x = \pm\sqrt{-1/3}$ and $x = \pm 1$

Solve

1. $x^4 - 5x^2 + 4 = 0$
2. $x^4 - 7x^2 - 18 = 0$
3. $x^4 - 9x^2 = 0$
4. $x^6 - 3x^3 + 2 = 0$
5. $(x + 3)^2 + (x + 3) = 42$
6. $x^6 - 5x^3 + 6 = 0$
7. $2x^4 + 7x^2 - 4 = 0$

166. As has been said the great importance of equations onsists in the fact that they may be used in the solution of oblems.

The problem to be solved involves certain numbers, in solving we must find the relation existing between these numbers and state it in algebraic language in the form of an equation. There are problems which are solved by other means than an equation; but those we consider here can all be stated in the form of equations.

For instance:—The relation existing between the volume and the three dimensions of a cubical block is expressed by the equation

$$v = abc \qquad (1)$$

The relation between the surface and the three dimensions of the same block is

$$S = 2\,(ab + ac + bc) \qquad (2)$$

These equations are perfectly general and apply to a cubical block of any dimensions. If any three of the numbers concerned in either equation are given the third can be found by solving the equation which is seen to be an equation of one variable. This solution may be found either before or after the fixed numbers are substituted.

We can thus express any one of the numbers in terms of the others, that is, we can solve the equation for any one of the numbers.

For instance, solve (1) and (2) for c

$$v = abc \quad \text{hence} \quad c = v/ab$$

$$S = 2\,(ab + ac + bc) \quad \text{hence} \quad c = \frac{S \;-\; 2ab}{2\,(a + b)}$$

Many of the laws of physics and geometry can be expressed or formulated in equations.

These general equations might be called formulas if we desire. The origin of the name is apparent.

The following equations are the statements of certain problems. They are given here that you may gain a little more practice in solving an equation for any letter.

1. The relation between the sides of a right angled triangle is given by the equation $a^2+b^2=c^2$, solve for each letter.

2. The area of a circle is given by $A=\pi r^2$ solve for r.

3. The area of a circular ring is $A=\pi R^2-\pi r^2$, solve for r.

4. $(r+m)\,a=r$ Solve for a , r.

5. $s-v=s/n$ Solve for n , s

6. $2S=n\,(a+t)$ Solve for t , n

7. $t=a+(n-1)\,d$ Solve for d , n

8. $2S=n\,(2a+(n-1)\,d)$ Solve for d , n

9. $m=n\,(1+kt)$ Solve for k

10. $S=\dfrac{rt-a}{r-1}$ Find r, t, a

11. $\dfrac{2R-h}{h}=\dfrac{h}{2R}$ Solve for h , R

12. $g-\dfrac{T}{M}=\dfrac{T}{m}-g$ Solve for each letter.

13. $n=1/6\left(\dfrac{a-h}{h}\right)$ Solve for h

14. $\dfrac{1}{f_1}+\dfrac{1}{f_2}=\dfrac{1}{F}$ Solve for each letter.

15. $R=\dfrac{rr_1}{r+r_1}$ Solve for r , r_1

16. $C=\dfrac{E}{R+r/N}$ Solve for each letter

17. $a=p\,\dfrac{n+1}{n-1}$ Solve for n

18. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ Solve for x , y

19. $x=\dfrac{2-z^2}{z^2+1}$ Solve for z

FRACTIONAL EQUATIONS.

167. If fractions of any sort appear in an equation it has been our practice to clear the equation of fractions by multiplying each member by the Lowest Common Denominator of the fractions.

As long as the variable does not appear in a denominator the equation is a simple, quadratic, cubic, etc., that is, it is an equation of the 1st, 2nd, 3rd, etc. degree.

But as soon as a variable appears in the denominator of any of the fractions the equation ceases to be an equation of any degree; it is then called a fractional equation.

In solving fractional equations, however, we proceed in the same way; we transform the equation into another free of fractions. This new equation will contain all the roots of the given equation; but in addition to these it may contain others also that have been introduced by the multiplication. How this may be done may be best seen by considering the effect on the roots of an equation of multiplying both members of the equation by a number containing the variable.

Take the equation $x + 1 = 3$ (1)

multiply by $x - 1$ $x^2 - 1 = 3x - 3$ (2)

transpose $x^2 - 3x + 2 = 0$

$$(x - 1)\ (x - 2) = 0$$

$$x = 1\ ,\ x = 2$$

2 is a root of both (1) and (2)

1 is a root of (2) but not of (1)

The root 1 has been introduced by the multiplier $x - 1$.

Such may be the case in fractional equations.

Substitution in the fractional equation will always sift out the true roots from the false.

Solve

1. $\frac{n+3}{4} = \frac{4}{n-3}$

2. $\frac{x}{3} = \frac{3}{x-1}$

3 $\frac{t}{2(1-t)} = \frac{1}{4}$

4. $\frac{3}{x} - \frac{1}{x} = \frac{5+(x-5)}{x}$

5. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = \frac{24}{x^2-4}$

6. $\frac{x+1}{x-1} = \frac{x+7}{x+3}$

7. $\frac{6}{y+2} - \frac{y+1}{y-2} + \frac{y^2}{y^2-4} = 0$

8. $\frac{1}{a-b} + \frac{a-b}{x} = \frac{1}{a+b} + \frac{a+b}{x}$

9. $\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}$

IRRATIONAL EQUATIONS.

168. Equations containing radicals or fractional exponents are called irrational equations.

The method of solving irrational equations is best shown by an example.

Take the equation $\sqrt{x+2} - x = 0$ (1)

transposing $\sqrt{x+2} = x$

squaring $x+2 = x^2$ (2)

whence $(x-2)\ (x+1) = 0$

$x = 2\ ,\ x = -1$

Substitute in (1)

2 is a root of (1), but -1 is not. A root has been induced by squaring.

The method may be stated in the following form.

1. Isolate one radical.
2. Clear of radicals by squaring both numbers.
3. Repeat 1 and 2 until all radicals have been removed.
4. Solve the resulting rational equation.
5. By substitution determine which of the roots of this equation are roots of the original irrational equation.

Solve

1. $\sqrt{x} - 5 = 0$
2. $n + \sqrt{n+6} = 14$
3. $\sqrt{5x+1} = 2 - \sqrt{x}$
4. $\sqrt{x+5} + \sqrt{x-4} = 9$
5. $2 = 3m + \sqrt{5m^2+11}$
6. $4s = \dfrac{s}{\sqrt{1-s^2}}$
7. $c = \sqrt{1 + \dfrac{1}{t^2}}$ Solve for t
8. $y = \sqrt{s^2 + c} - c$ Solve for c
9. $\dfrac{x}{4\sqrt{9+x^2}} - \dfrac{1}{5} = 0$
10. Solve for h, and find its value when $T = 32$ and $a = 2$

$$T = 2a\sqrt{\frac{1}{4}\ a^2 + h^2 + a^2}$$

11. $\sqrt{a-x} + \sqrt{b-x} = \dfrac{b}{\sqrt{b-x}}$
12. $t = \pi\sqrt{\dfrac{m}{g}}$

Solve for m ; find m when $t = 1$. $\pi = 22/7$, $g = 32.2$

160. The great importance of algebra lies in its usefulness in solving problems. In fact that is what algebra is, a method of solving problems by means of symbols. In your future study of geometry, physics, astronomy, and other sciences you will have continually to draw upon your knowledge of algebra to help you solve the problems that come up. It is often quite difficult to state the problem in algebraic form but it is very fascinating work when one puts ones mind to it. You must use your common sense and think carefully. You may find considerable that is interesting and valuable in the following problems. And it may be that they will suggest others that you would like to tackle.

In solving these problems put your work in the very best form you can.

PROBLEMS.

1. If five be subtracted from seven times a number the result will be seven more than three times the number. What is the number?

2. If 4 be added to 9 times a number the result will be 1111. What is the number?

3. Divide 60 into two parts proportional to 2 and 3.

4. The sum of the two digits of a number is 11; if 27 be added to the number the digits are reversed. What is the number?

5. Show that the sum of two consecutive odd numbers is divisible by 4.

6. Show that the difference between a number and the number formed by writing its digits in reverse order is divisible by 9.

7. Three pipes supply a certain cistern which can hold 720 gallons. One supplies a gallon a minute, another 3 gallons a

minute and a third 5 gallons a minute. How long will it take to fill the cistern with all running?

8. There are two faucets in the kitchen; a bucket can be filled in three minutes from the cold water faucet, and in 70 seconds from the hot water faucet. How long will it take to fill the bucket if both faucets run?

9. A man can walk 4 miles an hour, or ride 10 miles an hour; how long will it take him to travel 50 miles if he ride 2/3 of the time?

10. A man started to walk to a point 10 miles distant. Half an hour after he started he was picked up by an automobile and carried to his destination at the rate of 12 miles an hour. He reached his destination in one hour and ten minutes from the time he started. What was his average rate of walking?

11. A certain river boat that can travel at the rate of 9 miles an hour takes 5 hours and 30 minutes to make a trip of 20 miles up the river and back, allowing a stop of 30 minutes. What is the average rate of the current?

12. Another boat makes the same trip in four and a half hours. What is its rate?

13. The first boat is to take a pleasure party up the river to be gone just two hours; when must the captain turn back? How far up the river do they go?

14. How much time should the captain of the first boat allow for a trip 15 miles up river and back with a stop of a hour?

15. A man can row in still water one mile an hour faster than the current. It takes him 8 hours to make a round trip of 18 miles. What is the rate of the current?

16. What number is four times its reciprocal?

17. Find two numbers such that the sum of the numbers and the difference of their square is 11.

18. Find the height of some object of your own choice.

19. Find the height of a tree by means of its shadow.

20. Without leaving the room find the distance of some object outside of the building by the use of similar triangles.

21. The height of a certain flagstaff is unknown, but it is noticed that the flag rope which is three feet longer than the flagstaff, when stretched out just reaches the ground at a point 20 feet from the foot of the staff; what is the height of the flagstaff.

22. Find two numbers whose product is 36, and the sum of whose squares is 97.

23. What number is 4/9 of its reciprocal?

24. The sum of the squares of three consecutive even numbers is 760; what are the numbers?

25. Two numbers are in the ratio 2 to 3 and the sum of their squares is 400; find the numbers.

27. The number of square inches in the surface of a cube exceeds the number of inches in the sum of its edges by 288; find the edge and the volume.

28. A rectangular field contains 340 square rods, one side being 8 rods shorter than the other, find the dimensions.

29. The same number represents the area and the circumference of a certain circle. What is its radius?

30. It is desired to turn out a spherical ball whose surface shall be one square foot. What will be its diameter?

31. The diameter of the sun is about 109 times that of the earth; how much larger is the sun than the earth?

32. How large a square can you cut from a circular piece of tin of one foot diameter?

33. Find the ratio of the area of the square and the circle f the last problem, giving two decimal places.

34. It is desired to make an octagonal tabouret top out of a piece of quarter sawed oak by cutting off the corners. How far from the corner must each cut be made?

35. What is the height of a isosceles triangle whose base is b, and side is a?

36. Make a cylinder of cardboard of which the length shall be 4 inches and the area of the entire surface shall be 75 square inches.

37. When a body is moving faster every second we say that it is moving with accelerated motion. The increase of velocity each second is called acceleration.

In the following problems we will put $v =$ velocity, $a =$ acceleration, $t =$ time, $s =$ space passed over.

Suppose the velocity of a body is increased 2 ft. every second; what will be its velocity 5 seconds after it starts? 7 seconds?

38. An electric car starts from rest, receiving an increase of velocity of 3 feet a second. What will be its velocity at the end of 5 seconds? 7 seconds? 10 seconds?

39. Suppose v, the velocity of a body is increased by a feet a second, what will be its velocity at the end of t seconds after starting?

40. Six seconds after starting an electric car has a velocity of 18 feet a second. Find its acceleration.

41. How long will it take an electric car to attain a velocity of 15 miles an hour if the speed is increased 2 feet a second?

42. If s equal the distance the body moves over in accelerated motion, then it has been found that

$$s = \frac{1}{2} at^2$$

How far would the car of Problem 41 have moved?

43. Within 7 seconds after an electric car had started it had moved 100 feet. What was the acceleration given?

44. If I let go of this ball I hold in my hand it will drop to the floor. We say that the earth pulls it. You will find that it moves faster and faster every moment until it is stopped by the floor. It has been found that the velocity varies as the time of falling, that it increases by about 32.2 feet a second. What is the acceleration in the case of a falling body? It is usual to denote this acceleration by the letter g. So that for bodies falling from rest the formulae connecting velocity, time, acceleration and distance are

$$v = gt \quad , \quad s = \frac{1}{2} gt^2$$

where $g = 32.2$ feet a second.

45. What velocity will a ball have 1 second after it has started to fall? 2 seconds? 3 seconds? 4, 5, 6 seconds?

46. How long would it take it to acquire a velocity of 20 miles an hour? 60 miles?

47. How far will a body fall in 1 second? In 2 seconds? In 3, 4, 5 seconds?

48. How far would it fall in each second up to the sixth?

49. How long would it take a body to fall 100 feet?

50. With what velocity would the ball of the last problem strike the ground?

51. A ball dropped from the brow of a cliff is seen to strike the water in 4 seconds. How high is the cliff?

52. How could you measure the depth of a well by means of your watch? Would it be a very accurate method? Why?

53. Find the height of one corner of the tower of Bradley Hall by this method and compare the result with that obtained before.

54. A stone dropped into a well was heard to strike the bottom in 2 seconds. What was the depth of the well supposing sound to travel 1120 feet a second?

55. Suppose a street car is running at the rate of 25 feet a second, then $v = 25$ at any time. Now suppose the motorman gives it an acceleration of 2 feet a second. In 3 seconds the increase of velocity will be 3×2 and the actual velocity will be $v = 25 + 3 \times 2 = 31$.

So in general $v = u + at$, if u stand for the velocity at the start.

56. A body having an initial velocity of 10 feet a second receives an acceleration of 5 ft. a second, in how many seconds will it acquire a velocity of 50 feet a second?

57. What was the distance passed over by the body of the last problem?

The formula is $s = ut + \frac{1}{2} at^2$

58. A car was going at the rate of 39 feet a second. It was stopped in 5 seconds; what acceleration was applied? How? What does the sign mean?

59. In what distance was the car of Problem 57 stopped?

60. What changes must be made in the formulae of (55) and (57) to make them apply to the case of a body thrown vertically into the air?

61. A ball is thrown vertically into the air with a velocity of 40 feet a second; when will it be at a height of 16 feet?

62. How high will the same ball rise in 2 seconds? Where will it be in 3 seconds?

63. When will it reach its greatest height?

64. A body thrown vertically into the air returned in 5 seconds; what was the velocity with which it started? How high did it rise?

65. How hard must I throw a ball up so that a man on a scaffold 25 feet above me shall just be able to catch it?

66. With what velocity would a stone let fall from the top of the Washington monument in the City of Washington strike the ground?

67. If a ball starts to rise with a velocity of 64 feet a second when will it reach a height of 28 feet? If $u=80$ when will $s=64$?

68. From a balloon, which is ascending with a velocity of 32 feet a second, a stone is dropped reaching the ground in 17 seconds. How high was the balloon when the stone was dropped?

69. Find a number whose square increased by 13 is equal to four times the number.

70. An express train that travels at the average rate of 40 miles an hour starts from a certain station 50 minutes after a freight train, and overtakes the freight in two hours and five minutes. Find the rate of the freight.

71. Find two factors of 2268 whose sum is 99.

72. A capitalist loaned k dollars at a certain rate of interest and withdrew each year b dollars; at the end of two years there remained n dollars invested. At what rate was the money loaned?

73. What must be the diameter of a ball which is to have a volume of 2 cubic feet?

74. Iron weighs .24 pounds per cubic inch, a ball weighing 12 pounds is to be made; what must be is diameter?

75. The diameter of a certain spherical ball is 4 inches; what must be the diameter of a similar ball of twice its size?

76. Which is the better bargain, oranges two and three quarters inches in diameter at 30 cents a dozen or oranges three inches in diameter at 45 cents a dozen?

CHAPTER 12.

EQUATIONS OF TWO VARIABLES.

170. As we have already seen in Chapter 4 the solution of a problem is often much simpler if two variables are used instead of one. In such cases it is necessary to have two equations: and these two equations must be furnished by the conditions given in the problem.

An illustration in point is the following:

A man found that he could row four miles down stream in thirty minutes, while it took him two hours to row back.

In this problem the only numbers that are known are the distance rowed and the times up and down stream. The speed of the current and the rate at which the man can row are unknown. Can they be found?

Let $r =$ rate of current in miles per hour.
and $v =$ rate man in miles per hour in still water
then $v + r =$ rate down stream.
$v - r =$ rate up stream
$\frac{1}{2}(v + r) =$ distance down stream
$2(v - r) =$ distance up stream

hence
$$\frac{1}{2}(v + r) = 4$$
$$2(v - r) = 4$$

These two equations simplified are
$$v + r = 8$$
$$v - r = 2$$

from which we get by addition

$$2v = 10 \quad \text{that is} \quad v = 5$$

and by substitution

$$5 + r = 8 \quad \text{that is} \quad r = 3$$

This set of values $v = 5$, $r = 3$ satisfies both equations and has a clear meaning when applied to the given problem. And hence these are the required numbers.

171. In the case just considered, both equations in which the conditions of the problem are stated are of the first degree. But by no means is this always the case. The equations may be of any kind and degree and there thus arises, of course, a variety of cases of quite different aspect. The best and most scientific way of studying the solution of such sets of equations is to classify them and take up the cases in regular order.

There thus opens up before us a vast field for investigation in which we, in this course, can make only a start. We will consider two simple cases of great importance.

1st. Both equations are of the first degree.

2nd. One equation is of the first degree, the other of the second.

THE FIRST CASE.

172. The first case has been considered in Chapter 4, but it will be well to make a review of it.

Three methods were given, one graphic and two algebraic; the graphic, addition, and substitution methods.

It was seen that every equation of the first degree in two variables has for its graph a straight line. Two straight lines intersect each other in but one point. The graphic method consisted in determining the co-ordinates of the point of intersection of the graphs of the equations.

The addition method proceeds thus:—

$$2n - 3p = 5 \qquad (1)$$
$$4n + 2p = 9 \qquad (2)$$

To find n eliminate p

Mult. (1) by 2 $\quad 4n - 6p = 10$
Mult. (2) by 3 $\quad 12n + 6p = 27$
Add $\quad 16n = 37$
$n = 37/16$

Substitute in (1) $\quad 37/8 - 3p = 5$
whence $\quad p = 77/24$

Check

The substitution method is often the simplest.

$$n - r = 7 \qquad (1)$$
$$3r = 5\,(n + 1) \qquad (2)$$

From (1) $\quad n = r + 7 \qquad (3)$
Subs. in (2) $\quad 3r = 5\,(r + 7 + 1)$
$3r = 5r + 40$
$r = -20$
Subs. in (3) $\quad n = -20 + 7$
$= -13$

Check

173. In the following problems use the method which seems to be best adapted to the equations at hand.

1. $4x + 5y = 40$
   $6x - 7y = 2$

2. $20 = 7p + 2q$
   $55 = 20p + 7q$

3. $3A = 6$
   $5A + B = 5$

4. $5 + a + 2b = 0$
   $7 + 5a + b = 0$

5. The sum of two numbers is 200, their difference is 79: what are the numbers?

6. In what proportions should two kinds of coffee worth 20 cts. be combined to obtain a mixture worth 26 cents?

7. $4x + 3y = 6000$
$x + 2y = 2000$

8. $5A + 3B = 19$
$2A - B = 1$

9. $25 = \frac{5(a+t)}{2}$
$t = a + (5-1)2$

10 $\frac{3}{4}w - \frac{1}{2}R = 0$
$\frac{1}{2}w + \frac{3}{4}R = 100$

11. $a + \frac{5}{2}d = \frac{5}{6}$
$a + 7d = 16/3$

12. $4(x - 3y) = 8$
$(x + y) / (x - 2y) = 3$

13. $5(x - 2y) - (x - y) = -24$
$11(2x + 3y) + (2x - y) = 200$

14. The equation $y = mx + b$, being of the first degree in the two variables, x and y is the equation of a straight line. But as the constants m and b are not known the position of the line is not fixed. Determine m and b so that the graph shall pass through the two points 3, 2 and 4, 1

Could you determine the equation of any straight line drawn on a sheet of co-ordinate paper?

15. $9 = k - k'$
$9 = 19k' - 361k'$

16. $3\left(\frac{x}{2}-\frac{y}{3}\right)=5$

$4-3\left(\frac{x}{4}+\frac{y}{6}\right)=2$

17. The sum of two numbers is s and their difference is d: find the numbers.

In the following four examples solve for x and y.

18. $2x+y=2a$

$2x-y=2b$

19. $ax=by$

$x+y=c$

20. $ax+by=1$

$bx+ay=1$

21 $\frac{x}{a+b}+\frac{y}{a-b}=2a$

$(x-y)/4ab=1$

22. $x=3a+2b$ Find a and b in terms of x and y

$y=a-5b$

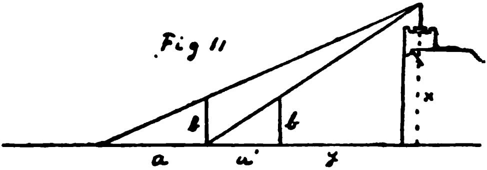
23. Suppose you were camping on a river. How could you find the rate of the current of the river and your own rate of rowing?

24. $180+(2p+2n)=R$

$180-(2p-2n)=R'$

Solve for p and n.

25. Find the height of the flagstaff on building we are in. This can be done by making the measurements of a, a′ and b as indicated in the figure, and then computing x and y, by means of the equations given below the figure.



$$\frac{a'}{a'+y}=\frac{b}{x} \qquad \frac{a}{a+a'+y}=\frac{b}{x}$$

Show that these equations are true.

The Second Case.

174. One equation of the first degree and the other of the second.

As an illustration of a problem giving a set of equations of this kind, we take the following:—

The sum of the squares of two numbers is 25, while twice one plus the other is 10. What are the numbers?

If x and y are the numbers then the equations are

$$x^2+y^2=25 \qquad (1)$$
$$2x+y=10 \qquad (2)$$

The most obvious plan to follow in solving this set of equations is to eliminate one of the letters by substitution

From (2) $y=10-2x \qquad (3)$

Substitute in (1) $x^2+(10-2x)^2=25$

which reduces to $x^2-8x+15=0$

whence $(x-3)\ (x-5)=0$

and $x=3$ and $x=5$

That is, x has two values. Substituting these in turn in equation (2) or more conveniently in (3), (why more conveniently?) we get two corresponding values for y.

when $x=3$, $y=10-\ \ 6=4$

when $x=5$, $y=10-10=0$

There are thus two solutions to the set of equations

$$x = 3\ ,\ y = 4 \quad \text{and} \quad x = 5\ ,\ y = 0$$

and as these are both sensible when applied to the problem, there are two solutions to the problem.

It is perfectly evident that the same method of substitution can be applied to every set of equations of this kind. We may state the method in the form of a rule.

1. Solve the linear equation for one of the variables.

2. Eliminate this variable by substituting its value in the quadratic equation.

3. Solve the resulting equation of one variable.

4. Find the corresponding value for the other variable by substituting the values just found in the linear equation.

5. Check by substituting in both the original equations.

Solve.

1. $2y - x = 5$
   $x^2 + y^2 = 25$

2. $x - y = 3$
   $xy = 40$

175. An interesting light is thrown upon the subject by drawing the graphs of the two equations just as was done in Case 1. Take the first example for illustration and draw the graphs with the same origin.

In drawing the graph of $x^2 + y^2 = 25$ assign to x integral values from -6 to $+6$ and compute the corresponding values for y. The work will be simplified if the equation be solved for y first. Thus:—

$$y = \pm\sqrt{25 - x^2}$$

The imaginary values cannot be plotted: only the real values will give points on the graph.

If your work has been carefully done it will be seen that the graph is a circle of radius 5, with its center at the origin.

The graph of the linear equation will intersect this graph in two points which correspond to the solutions of the set of equations as determined in the algebraic solution.

3. In the same manner plot the equations of example 2, and thus verify the solutions.

176. The graphs of the two quadratic equations $x^2 + y^2 = 25$ and $xy = 40$ are very unlike.

The graphs of all linear equations are alike, they are straight lines. But the graphs of quadratics are of several types. The following equations give all the types and you are asked to draw them carefully.

1. $x^2 + y^2 = 25$ already drawn.
2. $x^2 - y^2 = 25$
3. $y^2 = 4x$
4. $4x^2 + 9y^2 = 36$
5. $x^2 - xy - 2y^2 = 0$

Names have been given to these types. They are: 1. A circle. 2. An hyperbola. 3. A parabola. 4. An ellipse. 5. Two straight lines.

Equations of quite different form often give the same type of graph. Thus the equation $4xy = 25$ gives the same type as $x^2 - y^2 = 25$. In fact it is the same curve except that it is differently placed on the diagram. You can make one coincide with the other. This is rather a fascinating subject but we cannot go into it very deeply here. You will study it in detain at some future time.

If you take a straight line and lay it anywhere on these figures just drawn you will find that in some cases you will have two intersections, in some cases one intersection and in some cases no intersection. Can you place the line so that there will ever be more than two intersections?

This illustrates the fact that a set of equations of the kind we are considering can never have more than two solutions.

Now I wish to ask you a question that will probably cause you a good deal of thinking. What light does this last statement throw upon the reason for step 4 of the rule for solving a set of equations of this kind? Why not substitute in the quadratic equation?

177. Solve the following and draw the graphs in each case.

1. $y^2 = 4x$
   $2x + y = 4$

2. $x^2 - y^2 = 119$
   $x - y = 7$

3. $y = 5 - 2x$
   $x^2 + y^2 = 5$

4. $y + 2x = 5$
   $x^2 + y^2 = 5$

These last two examples suggest two questions:—

Can you find a short method for drawing the graph of the equation $x^2 + y^2 = 5$?

The graph of an equation of the form $x^2 + y^2 = a^2$ is a circle whose radius is a and whose center is the origin.

If the solutions of a set of equations contain imaginary numbers, what is true of the intersections of their graphs?

5. $xy - y = 5$
   $x + y = 3$

6. $4x^2 + y^2 = 4$
   $y + 2x = 1$

Solve, but do not plot graphs.

7. $45 = \frac{n(a+9)}{2}$
   $9 = a + (n - 1)$

8. $-\frac{3}{2} = \frac{n(1/3 + t)}{2}$

$t = \frac{1}{3} + (n-1)\ (-\frac{1}{12})$

9. The sum of two numbers is 27, their product is 180: what are the numbers?

10. The area of a rectangle is 84, while its perimenter is 38: what are its dimensions?

11. $x + y = b$
$xy = a$

12. $x^2 - y^2 = a$
$x - y = b$

13. $x/a = y/b$
$x^2 + y^2 = 1$

14. $a^2 + b^2 = h^2$
$b - a = h - b$

Solve for a and b.

15. Given $2as = y^2$ and $f = ma$: find an equation free from a.

16. Given $prt = i$ and $p + i = a$

Find a formula for computing the principal when amount, rate and time are given.

17. Find an equation showing the relation of the circumference and area of a circle.

By this formula compute the area of a circle of circumference two feet.

18. By the use of the same equation find how long a fence must be to enclose a circular lot of 200 square feet.

19. From $a = v/t$ and $v = ft/m$ eliminate t.

20. From $v = gt$ and $s = \frac{1}{2}gt^2$ eliminate t.

21. In Chapter 11 we learned that if a ball be thrown vertically upward the velocity at any instant and the distance from the earth are given by the formulae

$$v = u - gt \qquad (1)$$

$$s = ut - \frac{1}{2}gt^2 \qquad (2)$$

where $u =$ velocity at the start, $t =$ time, $v =$ velocity at time, t and $s =$ distance from earth and $g = 32.2$

Equation (1) connects v, u, and t.

Equation (2) conects s, u, and t.

A base ball thrown vertically into the air reached a height of 60 feet. What was the velocity with which it was thrown and how long was it rising? How long before it reached the ground.

22. By eliminating t from equations (1) and (2) find an equation connecting v, s, and u.

23. Solve for the two variables n and e

$$2s = n\,(a + l)$$
$$l = a\,(n - 1)d$$

Determine n and l when $s = 25$, $a = 1$, $d = 2$.

24. Find h in terms of e

$$s^2 + \left(\frac{e}{2}\right)^2 = e^2$$

$$h^2 + \left(\frac{s}{3}\right)^2 = s^2$$

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